1 Sphere

One has

$$
\begin{aligned}
\int \frac{dV}{2r} &= \frac{1}{2}\int_0^a r\,dr \int_{-1}^1 d\cos\theta \int_0^{2\pi} d\phi = \frac{1}{2}\frac{a^2}{2}\,2\cdot 2\pi = \pi a^2, \\
\int \frac{\cos^2\theta}{2r}\,dV &= \frac{\pi a^2}{2}\int_{-1}^1 \cos^2\theta\,d\cos\theta = \frac{1}{3}\,\pi a^2, \\
\int \frac{\cos^4\theta}{2r}\,dV &= \frac{\pi a^2}{2}\int_{-1}^1 \cos^4\theta\,d\cos\theta = \frac{1}{5}\,\pi a^2,
\end{aligned}
\tag{1}
$$

or

$$
\int \frac{\cos^{2n}\theta}{2r}\,dV = \frac{\pi a^2}{2n+1}. \tag{2}
$$

Next,

$$
(3\cos^2\theta - 1)\sin^2\theta = (3\cos^2\theta - 1)(1-\cos^2\theta) = 4\cos^2\theta - 3\cos^4\theta - 1. \tag{3}
$$

Consequently, a correction

$$
\frac{P}{2}\,k^2 r^2 \sin^2\theta, \tag{4}
$$

for **E** along the z axis of rotational symmetry and **k** along the x axis, modifies the dynamic depolarization term to

$$
k^2\,\frac{5\cos^2\theta - 3\cos^4\theta}{2r}, \tag{5}
$$

which results in

$$
k^2 \int \frac{5\cos^2\theta - 3\cos^4\theta}{2r}\,dV = \frac{16\pi}{15}\,x^2 = \frac{4\pi}{3}\,\frac{4}{5}\,x^2. \tag{6}
$$

On the other hand, a correction

$$
\mathbf{P}\,(\mathbf{k}\cdot\mathbf{r})^2 = \mathbf{P}\,k^2 r^2 \sin^2\theta\cos^2\phi, \tag{7}
$$

which assumes **E** along the z axis of rotational symmetry and **k** along the x axis, introduces a correction to the dynamic depolarization of

$$
\pi k^2 a^2 \left(\frac{4}{3} - \frac{3}{5} - 1\right) = \frac{\pi x^2}{15}\,(20 - 9 - 15) = -\frac{4\pi}{3}\,\frac{x^2}{5}. \tag{8}
$$

The factor of 1/2 comes form a fact that

$$
\int_0^{2\pi} \cos^2\phi\,d\phi = \int_0^{2\pi} \frac{1+\cos 2\phi}{2}\,d\phi = \pi = \frac{1}{2}\int_0^{2\pi} d\phi. \tag{9}
$$

2 Integration in cartesian coordinates

It is recommended to perform the volume integration in cartesian coordinates. Upon the substitution

$$
x = ax', \qquad y = by', \qquad z = cz', \tag{10}
$$

the ellipsoid

$$
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{11}
$$

is transformed into a unit sphere,

$$
(x')^2 + (y')^2 + (z')^2 = 1. \tag{12}
$$

The corresponding volume elements are related by

$$dV = abc\, dV'. \tag{13}$$

The radius vector square is

$$r^2 = a^2(x')^2 + b^2(y')^2 + c^2(z')^2. \tag{14}$$

For a spheroid with its axis of axial symmetry along the z-axis and with $a = b$, the volume integration can be performed over discs in the plane perpendicular to z and finally over z. Using cylindrical coordinates in the plane perpendicular to z,

$$\begin{aligned} r^2 &= a^2\rho^2 + c^2(z')^2 = a^2[\rho^2 + (c^2/a^2)(z')^2], \\ \cos^2\theta + 1 &= \frac{1}{r^2}(z^2 + r^2) = \frac{1}{r^2}[a^2\rho^2 + 2c^2(z')^2] = \frac{a^2}{r^2}[\rho^2 + 2(c^2/a^2)(z')^2] \\ dV &= a^2 b\, \rho\, dz' d\rho d\varphi, \end{aligned} \tag{15}$$

where the volume integral is taken over $z' \in (-1, 1)$, $\rho \in (0, \sqrt{1-(z')^2})$, $\varphi \in (0, 2\pi)$. Therefore, on using (87),

$$\begin{aligned} \int \frac{\cos^2\theta + 1}{r} dV &= 4\pi a^2 c \int_0^1 dz' \int_0^{\sqrt{1-(z')^2}} \frac{a^2\rho^2 + 2c^2(z')^2}{[a^2\rho^2 + c^2(z')^2]^{3/2}}\, \rho\, d\rho \\ &= 4\pi ac \int_0^1 dz' \int_0^{\sqrt{1-(z')^2}} \frac{\rho^2 + 2(c^2/a^2)(z')^2}{[\rho^2 + (c^2/a^2)(z')^2]^{3/2}}\, \rho\, d\rho \\ &= 4\pi ac \int_0^1 dz' \left. \frac{\rho^2}{[\rho^2 + (c^2/a^2)(z')^2]^{1/2}} \right|_{\rho=0}^{\rho=\sqrt{1-(z')^2}} \end{aligned} \tag{16}$$

On using $\cos\theta = z/r$ and (81) [or (82) for $n = 1$],

$$\begin{aligned} \int \frac{\cos^2\theta}{2r} dV &= 2\pi a^2 c \int_0^1 dz' \int_0^{\sqrt{1-(z')^2}} \frac{c^2(z')^2}{[a^2\rho^2 + c^2(z')^2]^{3/2}}\, \rho\, d\rho \\ &= 2\pi ac \int_0^1 dz' \int_0^{\sqrt{1-(z')^2}} \frac{(c^2/a^2)(z')^2}{[\rho^2 + (c^2/a^2)(z')^2]^{3/2}}\, \rho\, d\rho \\ &= -2\pi ac(c^2/a^2) \int_0^1 dz' \left. \frac{(z')^2}{[\rho^2 + (c^2/a^2)(z')^2]^{1/2}} \right|_{\rho=0}^{\rho=\sqrt{1-(z')^2}} \end{aligned} \tag{17}$$

On using (82) for $n = 2$,

$$\begin{aligned} \int \frac{\cos^4\theta}{2r} dV &= 2\pi a^2 c \int_0^1 dz' \int_0^{\sqrt{1-(z')^2}} \frac{c^4(z')^4}{[a^2\rho^2 + c^2(z')^2]^{5/2}}\, \rho\, d\rho \\ &= 2\pi ac(c^4/a^4) \int_0^1 dz' \int_0^{\sqrt{1-(z')^2}} \frac{(z')^4}{[\rho^2 + (c^2/a^2)(z')^2]^{5/2}}\, \rho\, d\rho \\ &= -\frac{2}{3}\pi ac(c^4/a^4) \int_0^1 \left. \frac{(z')^4\, dz'}{[\rho^2 + (c^2/a^2)(z')^2]^{3/2}} \right|_{\rho=0}^{\rho=\sqrt{1-(z')^2}} \end{aligned} \tag{18}$$

Here we have to differentiate in between prolate and oblate spheroids. In what follows, we define

$$g^2 = \frac{a^2}{c^2 - a^2} > 0, \tag{19}$$

for a prolate spheroid, and

$$g^2 = \frac{a^2}{a^2 - c^2} > 1, \tag{20}$$

for a oblate spheroid.

2.1 Prolate spheroid

For a prolate spheroid, the following integral results:

$$\begin{aligned}\int \frac{\cos^2\theta+1}{r}\,dV &= 4\pi ac\int_0^1 dz'\frac{1-(z')^2}{[1+(z')^2/g^2]^{1/2}}\\ &= 4\pi acg\int_0^1 \frac{1-(z')^2}{[g^2+(z')^2]^{1/2}}\,dz',\end{aligned} \tag{21}$$

Eventually, on using (76), (78),

$$\int \frac{\cos^2\theta+1}{r}\,dV = 4\pi acg\left[-\frac{\sqrt{1+g^2}}{2}+\left(1+\frac{g^2}{2}\right)\ln\frac{1+\sqrt{1+g^2}}{g}\right], \tag{22}$$

$$\begin{aligned}\int \frac{\cos^2\theta}{2r}\,dV &= -2\pi ac(c^2/a^2)g\int_0^1 \frac{(z')^2\,dz'}{[g^2+(z')^2]^{1/2}}+2\pi ac(c^2/a^2)\frac{a}{c}\int_0^1 z'\,dz'\\ &= \pi c^2-2\pi g(c^3/a)\left[\frac{\sqrt{1+g^2}}{2}-\frac{g^2}{2}\ln\frac{1+\sqrt{1+g^2}}{g}\right],\end{aligned} \tag{23}$$

$$\begin{aligned}\int \frac{\cos^4\theta}{2r}\,dV &= -\frac{2}{3}\pi ac(c^4/a^4)\int_0^1 \left.\frac{(z')^4\,dz'}{[\rho^2+(c^2/a^2)(z')^2]^{3/2}}\right|_{\rho=0}^{\rho=\sqrt{1-(z')^2}}\\ &= -\frac{2}{3}\pi ac(c^4/a^4)\int_0^1\left(\frac{(z')^4\,dz'}{[1+(z')^2/g^2]^{3/2}}-\frac{a^3z'}{c^3}\right)\\ &= \frac{a^3}{2c^3}\frac{2}{3}\pi ac(c^4/a^4)-\frac{2}{3}\pi ac(c^4/a^4)\int_0^1\frac{(z')^4\,dz'}{[1+(z')^2/g^2]^{3/2}}\\ &= \frac{\pi c^2}{3}-\frac{2}{3}\frac{\pi c^5}{a^3}\int_0^1\frac{(z')^4\,dz'}{[1+(z')^2/g^2]^{3/2}}\cdot\end{aligned} \tag{24}$$

Eventually, on using (84),

$$\begin{aligned}\int \frac{\cos^4\theta}{2r}\,dV &= \frac{\pi c^2}{3}-\frac{2}{3}\frac{\pi c^5}{a^3}g^3\left.\left(\frac{x^3}{2\sqrt{g^2+x^2}}+\frac{3g^2}{2}\frac{x}{\sqrt{g^2+x^2}}-\frac{3g^2}{2}\ln\left|x+\sqrt{g^2+x^2}\right|\right)\right|_0^1\\ &= \frac{\pi c^2}{3}-\frac{2}{3}\frac{\pi c^2}{e^3}\left(\frac{1}{2\sqrt{g^2+1}}+\frac{3g^2}{2\sqrt{g^2+1}}-\frac{3g^2}{2}\ln\frac{1+\sqrt{g^2+1}}{g}\right)\\ &= \frac{\pi c^2}{3}-\frac{2}{3}\frac{\pi c^2}{e^3}\left(\frac{e}{2}+\frac{3eg^2}{2}-\frac{3g^2}{2}\ln\frac{1+\sqrt{g^2+1}}{g}\right)\\ &= \frac{\pi c^2}{3}-\frac{\pi c^2}{e}\left(\frac{1}{3e}+\frac{g^2}{e}-\frac{g^2}{2e^2}\ln\frac{1+e}{1-e}\right)\end{aligned} \tag{25}$$

The definition of g should be contrasted with a conventional *eccentricity*, which is defined as the ratio of the difference of the squares of the major and minor semiaxis divided by the square of the major semiaxis (cf. Eq. (5.33) of [?]):

$$1\geq e^2=\frac{c^2-a^2}{c^2}>0, \tag{26}$$

and which values are always within the interval $(0,1)$. Note in passing that

$$1+g^2=\frac{c^2}{c^2-a^2}=\frac{1}{e^2},\qquad g^2=\frac{1-e^2}{e^2}\cdot \tag{27}$$

Therefore, the final result can be recast as

$$
\begin{aligned}
\int \frac{\cos^2\theta + 1}{r}\, dV &= 4\pi acg\left[-\frac{1}{2e} + \left(1 + \frac{1-e^2}{2e^2}\right)\ln\frac{1+e}{\sqrt{1-e^2}}\right] \\
&= 4\pi acg\left(-\frac{1}{2e} + \frac{1+e^2}{2e^2}\ln\frac{1+e}{\sqrt{1-e^2}}\right) \\
&= 4\pi ac\,\frac{\sqrt{1-e^2}}{2e^2}\left(-1 + \frac{1+e^2}{e}\ln\frac{1+e}{\sqrt{1-e^2}}\right) \\
&= 4\pi ac\,\frac{\sqrt{1-e^2}}{2e^2}\left(-1 + \frac{1+e^2}{2e}\ln\frac{1+e}{1-e}\right) \\
&= 2\pi c^2\left(L_> + \frac{1-e^2}{2e}\ln\frac{1+e}{1-e}\right) \\
&= 4\pi a^2\,\frac{1}{2e^2}\left(-1 + \frac{1+e^2}{2e}\ln\frac{1+e}{1-e}\right) \\
&= \frac{3V}{2c}\,\frac{1}{e^2}\left(-1 + \frac{1+e^2}{2e}\ln\frac{1+e}{1-e}\right) \\
&= \frac{3V}{2c}\left(\frac{1+e^2}{1-e^2}L_> + 1\right),
\end{aligned}
\tag{28}
$$

where we have used that

$$
\sqrt{1-e^2} = \frac{a}{c}, \qquad g = \frac{a}{c}\frac{1}{e}. \tag{29}
$$

Analogously,

$$
\begin{aligned}
\int \frac{\cos^2\theta}{2r}\, dV &= \pi c^2 - 2\pi g(c^3/a)\left[\frac{\sqrt{1+g^2}}{2} - \frac{g^2}{2}\ln\frac{1+\sqrt{1+g^2}}{g}\right] \\
&= \pi c^2 - 2\pi g(c^3/a)\left(\frac{1}{2e} - \frac{1-e^2}{2e^2}\ln\frac{1+e}{\sqrt{1-e^2}}\right) \\
&= \pi c^2 - 2\pi g(c^3/a)\left(\frac{1}{2e} - \frac{1-e^2}{4e^2}\ln\frac{1+e}{1-e}\right) \\
&= \pi c^2 - \frac{\pi c^2}{e^2}\left(1 - \frac{1-e^2}{2e}\ln\frac{1+e}{1-e}\right) \\
&= -\pi c^2\frac{1-e^2}{e^2} + \pi c^2\,\frac{1-e^2}{e^3}\,\frac{1}{2}\ln\frac{1+e}{1-e} \\
&= \pi c^2\frac{1-e^2}{e^3}\left(-e + \frac{1}{2}\ln\frac{1+e}{1-e}\right) \\
&= \pi c^2 L_> \\
&= \frac{\pi a^2}{1-e^2}\, L_>.
\end{aligned}
\tag{30}
$$

In the limiting case of a sphere one then obtains $\pi a^2/3$ as expected.

On comparing the last formula with the 5th line of the preceding formula,

$$
\begin{aligned}
\int \frac{dV}{2r} &= \pi c^2\frac{1-e^2}{2e}\ln\frac{1+e}{1-e} \\
&= \pi a^2\frac{1}{e}\,\mathrm{arctanh}\; e.
\end{aligned}
\tag{31}
$$

On inverting the defining equation (**??**) for L_z, one can also use

$$\frac{1}{e}\,\mathrm{arctanh}\, e = \frac{e^2}{1-e^2}\, L_z + 1. \tag{32}$$

Eventually, on applying the relations (27)

$$\begin{aligned}
\int \frac{\cos^4\theta}{2r}\, dV &= \frac{\pi c^2}{3} - \frac{\pi c^2}{e}\left(\frac{1}{3e} + \frac{1-e^2}{e^3} - \frac{1-e^2}{2e^4}\ln\frac{1+e}{1-e}\right) \\
&= \frac{\pi c^2}{3} - \frac{\pi c^2}{3e^2} + \frac{\pi c^2}{e^2}\,\frac{1-e^2}{e^3}\left(-e + \frac{1}{2}\ln\frac{1+e}{1-e}\right) \\
&= \frac{\pi c^2}{e^2}\left[L_> - \frac{1-e^2}{3}\right] \\
&= \frac{\pi a^2}{e^2}\left[\frac{L_>}{1-e^2} - \frac{1}{3}\right] \qquad (33)
\end{aligned}$$

Since

$$L_> \sim \frac{1}{3} - \frac{2}{15}\, e^2. \tag{34}$$

for $e \ll 1$, in the limiting case of a sphere one then obtains $\pi a^2/5$ as expected.

Consequently,

$$\begin{aligned}
\tilde{D}_\parallel &= 5\pi c^2 L_> - \frac{\pi c^2}{e^2}\,[3L_> + e^2 - 1] \\
&= 5\pi a^2 \frac{L_>}{1-e^2} - \frac{\pi a^2}{(1-e^2)e^2}\,[3L_> + e^2 - 1] \\
&= \frac{3V}{4c}\left\{\frac{(5e^2-3)}{(1-e^2)e^2}\, L_> + \frac{1}{e^2}\right\}, \qquad (35)
\end{aligned}$$

where we have used that $\pi a^2 = 3V/4c$.

For a comparison, the corresponding *geometrical factor* of a prolate spheroid would be (cf. Eq. (5.33) of [**?**])

$$L_z = L_> = \frac{abc}{2}\int_0^\infty \frac{dq}{(c^2+q)f(q)} = \frac{1-e^2}{e^2}\left(-1 + \frac{1}{2e}\,\ln\frac{1+e}{1-e}\right), \tag{36}$$

where

$$f(q) = [(q+a^2)(q+b^2)(q+c^2)]^{1/2}. \tag{37}$$

On using the limiting expressions

$$-1 + \frac{1+e^2}{2e}\ln\frac{1+e}{1-e} \sim e^2/3 + e^4/5 \qquad (e \to 0) \tag{38}$$

$$\begin{aligned}
\frac{e}{2}\,\ln\left(\frac{1+e}{1-e}\right) &= \frac{e}{2}\,[\ln(1+e) - \ln(1-e)] \\
&= e\sum_{l=0}^{\infty}\frac{e^{2l+1}}{2l+1} \sim e^2 + e^4/3 + e^6/5 \qquad (e \to 0) \qquad (39)
\end{aligned}$$

one finds in the limit $e \to 0$

$$k^2 \int \frac{\cos^2\theta + 1}{2r}\, dV = \frac{k^2 V}{c}\frac{3}{4}\left(\frac{1+e^2}{1-e^2}L_> + 1\right) \sim \frac{k^2 V}{c}\left(1 + \frac{2}{5}\, e^2\right), \tag{40}$$

and

$$L_> \sim \frac{1-e^2}{e^2}\,(e^2/3 + e^4/5) \sim (1-e^2)(1/3 + e^2/5) = 1/3 - (1/3 - 1/5)e^2 = \frac{1}{3}\left(1 - \frac{2}{5}\, e^2\right). \tag{41}$$

2.1.1 Perpendicular polarization

Upon denoting

$$\begin{aligned} L_x &= \int \frac{\cos^2\theta+1}{r}\,dV = ac\int \frac{2(x')^2+(y')^2+(c^2/a^2)(z')^2}{[\rho^2+(c^2/a^2)(z')^2]^{3/2}}\,dV', \\ L_y &= \int \frac{\cos^2\theta+1}{r}\,dV = ac\int \frac{2(y')^2+(x')^2+(c^2/a^2)(z')^2}{[\rho^2+(c^2/a^2)(z')^2]^{3/2}}\,dV', \end{aligned} \tag{42}$$

in the case of electric field applied perpendicular to the axis of axial symmetry, where obviously

$$L_x = L_y, \tag{43}$$

one has on using the formulas (77) and (74),

$$\begin{aligned} L_x+L_y+L_z &= 16\pi ac\int_0^1 dz'\int_0^{\sqrt{1-(z')^2}} \frac{\rho^2+(c^2/a^2)(z')^2}{[\rho^2+(c^2/a^2)(z')^2]^{3/2}}\,\rho\,d\rho \\ &= 16\pi ac\int_0^1 dz'\int_0^{\sqrt{1-(z')^2}} \frac{\rho\,d\rho}{[\rho^2+(c^2/a^2)(z')^2]^{1/2}} \\ &= 16\pi ac\int_0^1 dz'\,[\rho^2+(c^2/a^2)(z')^2]^{1/2}\Big|_0^{\sqrt{1-(z')^2}} \\ &= 16\pi ac\int_0^1 dz'\left\{[1+(z')^2/g^2]^{1/2}-\frac{c}{a}\,z'\right\} \\ &= -8\pi ac^2+\frac{16\pi ac}{g}\left\{\frac{1}{2}\sqrt{1+g^2}+\frac{g^2}{2}\ln\frac{1+\sqrt{1+g^2}}{g}\right\} \\ &= -8\pi ac^2+\frac{8\pi ac}{g}\left\{\sqrt{1+g^2}+g^2\ln\frac{1+\sqrt{1+g^2}}{g}\right\}. \end{aligned} \tag{44}$$

Now

$$\begin{aligned} \frac{\sqrt{1+g^2}}{g} &= \frac{1}{\sqrt{1-e^2}} = \frac{c}{a}, \\ g\ln\frac{1+\sqrt{1+g^2}}{g} &= \frac{\sqrt{1-e^2}}{e}\ln\frac{1+e}{\sqrt{1-e^2}} = \frac{a}{c}\,\frac{1}{2e}\ln\frac{1+e}{1-e}\cdot \end{aligned} \tag{45}$$

Therefore,

$$\begin{aligned} L_x+L_y+L_z &= -8\pi ac^2+8\pi ac^2+8\pi a^2\,\frac{1}{2e}\ln\frac{1+e}{1-e} \\ &= \frac{8\pi a^2}{e}\,\text{arctanh}\ e. \end{aligned} \tag{46}$$

On using

$$\text{arctanh}\ z \sim z+z^3/3+z^5/5+\mathcal{O}(z^7), \tag{47}$$

$$L_x+L_y+L_z \sim 8\pi a^2\,\left[1+e^2/3+e^4/5+\mathcal{O}(z^7)\right] \tag{48}$$

0n using the formulas (81), (85)

$$L_x+L_y = 4\pi ac\int_0^1 dz'\int_0^{\sqrt{1-(z')^2}} \frac{3\rho^2+2(c^2/a^2)(z')^2}{[\rho^2+(c^2/a^2)(z')^2]^{3/2}}\,rho\,d\rho$$

$$= \quad 4\pi ac \int_0^1 dz' \, \frac{3\rho^2 + 4(c^2/a^2)(z')^2}{[\rho^2 + (c^2/a^2)(z')^2]^{1/2}} \Bigg|_{\rho=0}^{\rho=\sqrt{1-(z')^2}}$$

$$= \quad 4\pi ac \int_0^1 dz' \left\{ \frac{3 + [4(c^2/a^2) - 3](z')^2}{[1 + [(c^2/a^2) - 1](z')^2]^{1/2}} - 4\frac{a}{c}\,|z'| \right\}$$

$$= \quad 4\pi ac \int_0^1 dz' \left\{ g\, \frac{3 + [4(c^2/a^2) - 3](z')^2}{[g^2 + (z')^2]^{1/2}} - 4\frac{a}{c}\,|z'| \right\}$$

$$= \quad 4\pi ac \left\{ \left[3 - \left(\frac{4c^2}{a^2} - 3\right)\frac{g^2}{2} \right] g \, \ln \frac{1 + \sqrt{1+g^2}}{g} + \left(\frac{4c^2}{a^2} - 3\right) \frac{g\,\sqrt{1+g^2}}{2} - 2\frac{a}{c} \right\} \quad (49)$$

2.2 Oblate spheroid

For an oblate spheroid, the following integral results:

$$\begin{aligned} \int \frac{\cos^2\theta + 1}{r}\, dV &= 4\pi ac \int_0^1 dz' \frac{1 - (z')^2}{[1 - (z')^2/g^2]^{1/2}} \\ &= 4\pi acg \int_0^1 \frac{1 - (z')^2}{[g^2 - (z')^2]^{1/2}}\, dz'. \end{aligned} \quad (50)$$

Eventually, on using (94),

$$\int \frac{\cos^2\theta + 1}{r}\, dV = 4\pi acg \left[\left(1 - \frac{g^2}{2}\right) \arcsin\frac{1}{g} + \frac{\sqrt{g^2-1}}{2} \right]. \quad (51)$$

Similarly,

$$\begin{aligned} \int \frac{\cos^2\theta}{2r}\, dV &= -2\pi(c^3/a) \int_0^1 dz' \frac{(z')^2}{[1 - (z')^2/g^2]^{1/2}} + 2\pi c^2 \int_0^1 z'\, dz' \\ &= \pi c^2 - 2\pi g(c^3/a) \int_0^1 \frac{1 - (z')^2}{[g^2 - (z')^2]^{1/2}}\, dz', \end{aligned} \quad (52)$$

and, eventually, on using (91),

$$\int \frac{\cos^2\theta}{2r}\, dV = \pi c^2 - 2\pi g(c^3/a) \left[-\frac{\sqrt{g^2-1}}{2} + \frac{g^2}{2} \arcsin\frac{1}{g} \right]. \quad (53)$$

On the other hand,

$$\begin{aligned} \int \frac{\cos^4\theta}{2r}\, dV &= -\frac{2}{3}\,\pi ac(c^4/a^4) \int_0^1 \frac{(z')^4\, dz'}{[\rho^2 + (c^2/a^2)(z')^2]^{3/2}} \Bigg|_{\rho=0}^{\rho=\sqrt{1-(z')^2}} \\ &= -\frac{2}{3}\,\pi(c^5/a^3) \int_0^1 dz' \left\{ \frac{(z')^4}{[1 - (z')^2/g^2]^{3/2}} - (a^3/c^3) z' \right\} \\ &= \frac{\pi c^2}{3} - \frac{2}{3}\,\pi g^3 (c^5/a^3) \int_0^1 dz' \, \frac{(z')^4}{[g^2 - (z')^2]^{3/2}} \\ &= \frac{\pi c^2}{3} - \frac{2}{3}\,\pi g^3 (c^5/a^3) \left\{ \frac{x^3}{2\sqrt{g^2-x^2}} - \frac{3g^2}{2} \frac{x}{\sqrt{g^2-x^2}} + \frac{3g^2}{2} \arcsin\frac{x}{g} \right\} \Bigg|_0^1 \\ &= \frac{\pi c^2}{3} + \frac{1}{3}\,\pi g^3 (c^5/a^3) \left\{ \frac{1}{\sqrt{g^2-1}} - \frac{3g^2}{\sqrt{g^2-1}} + 3g^2 \, \arcsin\frac{1}{g} \right\} \end{aligned} \quad (54)$$

where we have applied the identity (93) in the final step.

g should be contrasted with a conventional *eccentricity* e that is defined in the present case by:

$$1 \geq e^2 = \frac{a^2 - c^2}{a^2} = \frac{1}{g^2} > 0, \qquad \frac{c}{a} = \sqrt{1-e^2}, \tag{55}$$

and

$$\sqrt{g^2 - 1} = \frac{\sqrt{1-e^2}}{e}\cdot \tag{56}$$

Therefore,

$$\begin{aligned}
\int \frac{\cos^2\theta}{2r}\,dV &= \pi c^2 - 2\pi c^2 \frac{\sqrt{1-e^2}}{e}\left(\frac{1}{2e^2}\arcsin e - \frac{\sqrt{1-e^2}}{2e}\right) \\
&= \pi c^2 - \frac{\pi c^2}{e^2}\left(\frac{\sqrt{1-e^2}}{e}\arcsin e - 1 + e^2\right) \\
&= \pi c^2\, L_< \\
&= \pi a^2\,(1-e^2) L_<,
\end{aligned} \tag{57}$$

$$\begin{aligned}
\int \frac{\cos^4\theta}{2r}\,dV &= \frac{\pi c^2}{3} + \frac{\pi c^2}{3}\,\frac{(1-e^2)^{3/2}}{e^3}\left\{\frac{e}{\sqrt{1-e^2}} - \frac{3}{e\sqrt{1-e^2}} + \frac{3}{e^2}\,\arcsin e\right\} \\
&= \frac{\pi c^2}{3} + \frac{\pi c^2}{3}\,\frac{1-e^2}{e^2}\left\{1 - \frac{3}{e^2} + \frac{3\sqrt{1-e^2}}{e^3}\,\arcsin e\right\} \\
&= \frac{\pi c^2}{3e^2} - \pi c^2\,\frac{1-e^2}{e^4}\left\{1 - \frac{\sqrt{1-e^2}}{e}\,\arcsin e\right\} \\
&= \frac{\pi c^2}{3e^2} - \pi c^2\,\frac{1-e^2}{e^2}\,L_< \\
&= \pi a^2\,\frac{1-e^2}{3e^2}\left[1 - 3(1-e^2)L_<\right]
\end{aligned} \tag{58}$$

Consequently,

$$\begin{aligned}
\tilde{D}_{\|} &= \pi a^2\left\{5(1-e^2)L_< - \frac{1-e^2}{e^2}\left[1 - 3(1-e^2)L_<\right]\right\} \\
&= \pi a^2\,\frac{1-e^2}{e^2}\left[5e^2 L_< - 1 + 3(1-e^2)L_<\right] \\
&= \frac{3V}{4c}\,\frac{1-e^2}{e^2}\left[(2e^2+3)L_< - 1\right],
\end{aligned} \tag{59}$$

where we have used that $\pi a^2 = 3V/4c$.

$$\begin{aligned}
\int \frac{\cos^2\theta + 1}{r}\,dV &= \frac{4\pi ac}{e}\left[\left(1 - \frac{1}{2e^2}\right)\arcsin e + \frac{\sqrt{(1/e^2) - 1}}{2}\right] \\
&= 4\pi ac\,\frac{\sqrt{1-e^2}}{2e^2}\left(1 - \frac{\sqrt{1-e^2}}{e}\arcsin e + \frac{e}{\sqrt{1-e^2}}\arcsin e\right) \\
&= 2\pi c^2\,\frac{1}{e^2}\left(1 - \frac{\sqrt{1-e^2}}{e}\arcsin e + \frac{e}{\sqrt{1-e^2}}\arcsin e\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{3Vc}{2a^2}\left(L_< + \frac{1}{e\sqrt{1-e^2}}\arcsin e\right) \\
&= 2\pi a^2\,\frac{1-e^2}{e^2}\left(1 - \frac{\sqrt{1-e^2}}{e}\arcsin e + \frac{e}{\sqrt{1-e^2}}\arcsin e\right) \\
&= \frac{3V}{2c}\,\frac{1-e^2}{e^2}\left(1 - \frac{\sqrt{1-e^2}}{e}\arcsin e + \frac{e}{\sqrt{1-e^2}}\arcsin e\right) \\
&= \frac{3V}{2c}\left[(1-e^2)L_< + \frac{\sqrt{1-e^2}}{e}\arcsin e\right] \\
&= \frac{3V}{2c}\left[1 + (1-e^2)L_< - e^2 L_<\right] \\
&= \frac{3V}{2c}\left[(1-2e^2)L_< + 1\right], \qquad (60)
\end{aligned}
$$

and

$$
\begin{aligned}
\int \frac{dV}{2r} &= \frac{\pi c^2}{e\sqrt{1-e^2}}\arcsin e \\
&= \pi a^2\,\frac{\sqrt{1-e^2}}{e}\arcsin e \qquad (61)
\end{aligned}
$$

where we have used that

$$
1 - e^2 = \frac{c^2}{a^2}. \qquad (62)
$$

and

$$
L_< = \frac{1}{e^2}\left(1 - \frac{\sqrt{1-e^2}}{e}\arcsin e\right). \qquad (63)
$$

Note in passing that

$$
L_< \sim \frac{1}{3} + \frac{2e^2}{15} \sim \frac{1}{3}\left(1 + \frac{2e^2}{5}\right) \qquad (e \to 0) \qquad (64)
$$

and one finds in the limit $e \to 0$

$$
k^2 \int \frac{\cos^2\theta + 1}{2r}\,dV = \frac{k^2 V}{c}\,\frac{3}{4}\left[(1-2e^2)L_< + 1\right] \sim \frac{k^2 V}{c}\left(1 - \frac{2}{5}e^2\right). \qquad (65)
$$

Note that in an *oblate* spheroid case the spheroid eccentricity as defined by Landau and Lifshitz [**?**] is in the range $0 < e < \infty$, whereas that as defined Bohren and Huffman [**?**], and used by a majority of other authors, has values within a finite range $0 < e_k < 1$, which is the same range as in the *prolate* spheroid case. Then the shape of the oblate spheroid ranges from a *disk* ($e = 1$) to a *sphere* ($e = 0$); that of the prolate spheroid ranges from a *needle* ($e = 1$) to a *sphere* ($e = 0$).

2.2.1 Perpendicular polarization

One has on using the formulas (77), (75)

$$
\begin{aligned}
L_x + L_y + L_z &= 16\pi ac \int_0^1 dz'\,\left[\rho^2 + (c^2/a^2)(z')^2\right]^{1/2}\Big|_0^{\sqrt{1-(z')^2}} \\
&= 16\pi ac \int_0^1 dz'\left\{[1-(z')^2/g^2]^{1/2} - \frac{c}{a}z'\right\} \\
&= -8\pi c^2 + \frac{16\pi ac}{g}\left\{\frac{\sqrt{g^2-1}}{2} + \frac{g^2}{2}\arcsin\frac{1}{g}\right\}
\end{aligned}
$$

$$= \quad -8\pi c^2 + \frac{8\pi ac}{g}\left\{\sqrt{g^2-1} + g^2\,\arcsin\frac{1}{g}\right\}. \tag{66}$$

Now

$$\begin{aligned}
\frac{\sqrt{g^2-1}}{g} &= e\sqrt{(1/e^2)-1} = \sqrt{1-e^2} = \frac{c}{a}, \\
g\,\arcsin\frac{1}{g} &= \frac{1}{e}\,\arcsin e.
\end{aligned} \tag{67}$$

Therefore,

$$\begin{aligned}
L_x + L_y + L_z &= -8\pi c^2 + 8\pi c^2 + \frac{8\pi ac}{e}\,\arcsin e \\
&= \frac{8\pi ac}{e}\,\arcsin e = 8\pi a^2\,\frac{\sqrt{1-e^2}}{e}\,\arcsin e.
\end{aligned} \tag{68}$$

On using (69),

$$\arcsin z \sim z + \frac{1}{6}\,z^3 + \frac{3}{40}\,z^5 + \mathcal{O}(z^7) \tag{69}$$

$$\begin{aligned}
\frac{\sqrt{1-e^2}}{e}\,\arcsin e &\sim (1-z^2/2-z^4/8)(1+e^2/6+3e^4/40) \sim 1 - \frac{1}{3}\,e^2 + \frac{(9-15-10)}{120}\,e^4 \\
&\sim 1 - \frac{1}{3}\,e^2 - \frac{2}{15}\,e^4,
\end{aligned} \tag{70}$$

$$L_x + L_y + L_z \sim 8\pi ac\left[1 + \frac{1}{6}\,e^2 + \frac{3}{40}\,e^4 + \mathcal{O}(e^6)\right] \tag{71}$$

$$L_x + L_y + L_z \sim 8\pi a^2\left[1 - \frac{1}{3}\,e^2 - \frac{2}{15}\,e^4 + \mathcal{O}(e^6)\right] \tag{72}$$

Since $a \to c$ as $e \to 0$, irrespective if spheroid is prolate or oblate,

$$L_x + L_y + L_z \to 8\pi a^2 \qquad\qquad (e \to 0). \tag{73}$$

3 Summary of elementary quadrature formulas

$$\int \sqrt{a^2+x^2}\,dx = \frac{1}{2}\,x\sqrt{a^2+x^2} + \frac{a^2}{2}\,\ln\left|x+\sqrt{a^2+x^2}\right|, \tag{74}$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}\,x\sqrt{a^2-x^2} + \frac{a^2}{2}\,\arcsin\frac{x}{|a|} \tag{75}$$

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left|x+\sqrt{a^2+x^2}\right|, \tag{76}$$

$$\int \frac{x\,dx}{\sqrt{a^2+x^2}} = \sqrt{a^2+x^2}, \tag{77}$$

$$\int \frac{x^2\,dx}{\sqrt{a^2+x^2}} = \frac{x\sqrt{a^2+x^2}}{2} - \frac{a^2}{2}\,\ln\left|x+\sqrt{a^2+x^2}\right| \tag{78}$$

$$\int \frac{x^4\,dx}{\sqrt{a^2+x^2}} = \frac{\sqrt{a^2+x^2}}{4}\left(x^3 - \frac{3}{2}\,a^2x\right) + \frac{3}{8}\,a^4\,\ln\left|x+\sqrt{a^2+x^2}\right| \tag{79}$$

$$\int \frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}} \tag{80}$$

$$\int \frac{x\,dx}{(a^2+x^2)^{3/2}} = -\frac{1}{\sqrt{a^2+x^2}}, \tag{81}$$

$$\int \frac{x\,dx}{(a^2+x^2)^{n+1/2}} = -\frac{1}{(2n-1)}\frac{1}{(a^2+x^2)^{n-1/2}}, \tag{82}$$

$$\int \frac{x^2\,dx}{(a^2+x^2)^{3/2}} = -\frac{x}{\sqrt{a^2+x^2}} + \ln\left|x+\sqrt{a^2+x^2}\right| \tag{83}$$

$$\begin{aligned}\int \frac{x^4\,dx}{(a^2+x^2)^{3/2}} &= \frac{x^3}{2\sqrt{a^2+x^2}} - \frac{3a^2}{2}\int \frac{x^2\,dx}{(a^2+x^2)^{3/2}} \\ &= \frac{x^3}{2\sqrt{a^2+x^2}} + \frac{3a^2}{2}\frac{x}{\sqrt{a^2+x^2}} - \frac{3a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right| \end{aligned} \tag{84}$$

$$\int \frac{x^3\,dx}{(a^2+x^2)^{3/2}} = \frac{x^2+2a^2}{\sqrt{a^2+x^2}} \tag{85}$$

$$\begin{aligned}\int \frac{x^{2m}\,dx}{\sqrt{a^2+x^2}} &= -\frac{\sqrt{a^2+x^2}}{2m}\left[x^{2m-1} + \sum_{k=1}^{m-1} \frac{(-1)^k(2m-1)(2m-3)\dots(2m-2k+1)}{2^k(m-1)(m-2)\dots(m-k)}\,a^{2k}x^{2m-2k-1}\right] \\ &\quad +(-1)^m a^{2m}\frac{(2m-1)!!}{2^m m!}\ln\left|x+\sqrt{a^2+x^2}\right| \end{aligned} \tag{86}$$

The integrals (81), (85) imply

$$\int \frac{2a^2+x^2}{(a^2+x^2)^{3/2}}\,x\,dx = \frac{x^2}{\sqrt{a^2+x^2}}. \tag{87}$$

The formulas can be easily verified by differentiation.

Here (74), (75) are given as (1.2.41.8), (1.2.46.8) by [?].

The integrals (76), (77), (78), (79), (80), (81), (83), (85), (86), which are given as (1.2.43.13), (1.2.43.14), (1.2.43.15), (1.2.43.6) (for $m=2$), (1.2.43.17-20), and (1.2.43.6), respectively, by [?], are related by the recurrence

$$\int \frac{x^m\,dx}{(x^2\pm a^2)^{n+1/2}} = \frac{x^{m-1}}{(m-2n)(x^2\pm a^2)^{n-1/2}} \mp \frac{(m-1)a^2}{m-2n}\int \frac{x^{m-2}\,dx}{(x^2\pm a^2)^{n+1/2}} \tag{88}$$

which is given as (1.2.43.3) by [?].

$$\int \frac{dx}{\sqrt{a^2-x^2}} = -i\ln\left|ix+\sqrt{a^2-x^2}\right| = \arcsin\frac{x}{|a|}, \tag{89}$$

$$\int \frac{x\,dx}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2} \tag{90}$$

$$\int \frac{x^2\,dx}{\sqrt{a^2-x^2}} = -\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\arcsin\frac{x}{|a|} \tag{91}$$

$$\int \frac{x^2\,dx}{(a^2-x^2)^{3/2}} = \frac{x}{\sqrt{a^2-x^2}} - \arcsin\frac{x}{|a|} \tag{92}$$

$$\begin{aligned}\int \frac{x^4\,dx}{(a^2-x^2)^{3/2}} &= -\frac{x^3}{2\sqrt{a^2-x^2}} + \frac{3a^2}{2}\int \frac{x^2\,dx}{(a^2-x^2)^{3/2}} \\ &= -\frac{x^3}{2\sqrt{a^2-x^2}} + \frac{3a^2}{2}\frac{x}{\sqrt{a^2-x^2}} - \frac{3a^2}{2}\arcsin\frac{x}{|a|} \end{aligned} \tag{93}$$

The integrals (89), (91), which are given as (1.2.48.12) and (1.2.48.14) by [?], imply

$$\int_0^1 \frac{1-x^2}{(a^2-x^2)^{3/2}}\,dx = \left(1-\frac{a^2}{2}\right)\arcsin\frac{1}{|a|} + \frac{\sqrt{a^2-1}}{2}. \tag{94}$$

$$\sqrt{1-z^2} \sim 1 - z^2/2 - z^4/8 + \mathcal{O}(z^6) \tag{95}$$

4 Elliptic coordinates

The volume integration in the elliptic coordinates turns out to be much more involved than in the cartesian coordinates.

Let a and b be principal major and minor spheroid semiaxis. Then the most common definition of prolate spheroidal coordinates (μ, θ, φ) is

$$\begin{aligned} x &= q \sinh\mu \, \sin\theta \, \cos\varphi \\ y &= q \sinh\mu \, \sin\theta \, \sin\varphi \\ z &= q \cosh\mu \, \cos\theta. \end{aligned} \tag{96}$$

The constant q (eccentrity) is determined from the relations

$$a^2 - b^2 = q^2, \qquad \frac{b}{a} = \tanh\mu. \tag{97}$$

An infinitesimal volume element:

$$\begin{aligned} dV &= q^3 \sinh\mu \, \sin\theta (\sinh^2\mu + \sin^2\theta) \, d\mu \, d\theta \, d\varphi \\ &= q^3 (\cos^2\theta - \cosh^2\mu) \, d[\cosh\mu] \, d[\cos\theta] \, d\varphi, \end{aligned} \tag{98}$$

where it is reminded here that

$$\cosh^2\mu - \sinh^2\mu = 1, \qquad \cosh'\mu = \sinh\mu, \qquad \sinh'\mu = \cosh\mu. \tag{99}$$

One has

$$r^2 = x^2 + y^2 + z^2 = q^2(\sinh^2\mu \sin^2\theta + \cosh^2\mu \cos^2\theta) = q^2(\sinh^2\mu + \cos^2\theta). \tag{100}$$

Meier and Wokaun [**?**] terms in prolate spheroidal coordinates (μ, θ, φ) are

$$\begin{aligned} \frac{dV}{r^3}(3\cos^2\theta - 1) &= \frac{3\cos^2\theta - 1}{(\sinh^2\mu + \cos^2\theta)^{3/2}} (\cos^2\theta - \cosh^2\mu) \, d[\cosh\mu] \, d[\cos\theta] \, d\varphi \\ \frac{k^2}{2r}(\cos^2\theta + 1) \, dV &= \frac{(qk)^2}{2} \frac{\cos^2\theta + 1}{(\sinh^2\mu + \cos^2\theta)^{1/2}} (\cos^2\theta - \cosh^2\mu) \, d[\cosh\mu] \, d[\cos\theta] \, d\varphi. \end{aligned} \tag{101}$$

Let us focus on the product

$$\begin{aligned} P &= (\cos^2\theta + 1) \sinh\mu \, \sin\theta (\sinh^2\mu + \sin^2\theta) \, d\mu \, d\theta \, d\varphi \\ &= -(\cos^2\theta + 1)(\sinh^2\mu + 1 - \cos^2\theta) \sinh\mu \, d\mu \, d\varphi \, d[\cos\theta] \\ &= -\left[\sinh^2\mu (\cos^2\theta + 1) + 1 - \cos^4\theta\right] \sinh\mu \, d\mu \, d\varphi \, d[\cos\theta] \\ &= -\left[1 + \sinh^2\mu + \sinh^2\mu \, \cos^2\theta - \cos^4\theta\right] \sinh\mu \, d\mu \, d\varphi \, d[\cos\theta] \end{aligned} \tag{102}$$

Therefore, the required integrals to perform the volume integral of the 2nd Meier and Wokaun [**?**] term are

$$\begin{aligned} I_0 &= \int_{-1}^{1} \frac{dx}{\sqrt{\sinh^2\mu + x^2}} = \ln \frac{\sqrt{\sinh^2\mu + 1} + 1}{\sqrt{\sinh^2\mu + 1} - 1} = \ln \frac{\cosh\mu + 1}{\cosh\mu - 1}, \\ I_2 &= \int_{-1}^{1} \frac{x^2 \, dx}{\sqrt{\sinh^2\mu + x^2}} = \cosh\mu - \frac{\sinh^2\mu}{2} \ln \frac{\cosh\mu + 1}{\cosh\mu - 1}, \\ &= \cosh\mu - \frac{\cosh^2\mu - 1}{2} \ln \frac{\cosh\mu + 1}{\cosh\mu - 1}, \\ I_4 &= \int_{-1}^{1} \frac{x^4 \, dx}{\sqrt{\sinh^2\mu + x^2}} = \frac{\cosh\mu}{4}(2 - 3\sinh^2\mu) + \frac{3}{8} \sinh^4\mu \ln \frac{\cosh\mu + 1}{\cosh\mu - 1} \\ &= \frac{\cosh\mu}{4}(5 - 3\cosh^2\mu) + \frac{3}{8}(\cosh^2 - 1)^2\mu \ln \frac{\cosh\mu + 1}{\cosh\mu - 1}, \end{aligned} \tag{103}$$

and

$$
\begin{aligned}
\int \frac{k^2}{2r}\,(\cos^2\theta + 1)\,dV &= \pi(qk)^2 \int \left[I_0(1+\sinh^2\mu) + \sinh^2\mu\, I_2 - I_4\right] \sinh\mu\, d\mu \\
&= \pi(qk)^2 \int \left[I_0 \cosh^2\mu + (\cosh^2\mu - 1)\, I_2 - I_4\right] \, d\cosh\mu \\
&= \pi(qk)^2 \int \left[-I_2 - I_4 + \cosh^2\mu(I_0 + I_2)\right] \, d\cosh\mu. \qquad (104)
\end{aligned}
$$

The integration range is here $\mu \in (0, (b/a))$.

References