A supplementary material to the paper Depolarization field of spheroidal particles

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1 Summary of elementary formulas

In the case of D_z, the integration over the ρ coordinate can be performed on using the following identities:

$$\int \frac{x\,dx}{(a^2+x^2)^{n+1/2}} = -\frac{1}{(2n-1)}\frac{1}{(a^2+x^2)^{n-1/2}}, \tag{1}$$

$$\int \frac{x^3\,dx}{(a^2+x^2)^{3/2}} = \frac{x^2+2a^2}{\sqrt{a^2+x^2}}, \tag{2}$$

which imply

$$\int \frac{2a^2+x^2}{(a^2+x^2)^{3/2}}\,x\,dx = \frac{x^2}{\sqrt{a^2+x^2}}. \tag{3}$$

The respective integrals (1) and (2) are given as (1.2.43.12) and (1.2.43.20) by [1]. In the case of the sum $D_x+D_y+D_z$, the ρ-integration can be performed on using the elementary identity:

$$\int \frac{x\,dx}{\sqrt{a^2+x^2}} = \sqrt{a^2+x^2}. \tag{4}$$

All the integrals here and below can be easily verified by differentiation. In arriving at final results, one makes repeated use of the following formula

$$\sqrt{1-e^2} = \begin{cases} \frac{a}{c}, & \text{prolate} \\ \frac{c}{a}, & \text{oblate} \end{cases} \tag{5}$$

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1.1 Quadrature formulas for a prolate spheroid

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left|x+\sqrt{a^2+x^2}\right|, \tag{6}$$

$$\int \frac{x^2\,dx}{\sqrt{a^2+x^2}} = \frac{x\sqrt{a^2+x^2}}{2} - \frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right|. \tag{7}$$

The respective integrals (6) and (7) are given as (1.2.43.13) and (1.2.43.15) by [1].

1.2 Quadrature formulas for an oblate spheroid

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{|a|}, \tag{8}$$

$$\int \frac{x^2\,dx}{\sqrt{a^2-x^2}} = -\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\arcsin\frac{x}{|a|}\cdot \tag{9}$$

The respective integrals (8) and (9) are given as (1.2.48.12) and (1.2.48.14) by [1].

1.3 Quadrature formulas for the sum rule

$$\int \sqrt{a^2+x^2}\,dx = \frac{1}{2}x\sqrt{a^2+x^2} + \frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right|, \tag{10}$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{|a|}\cdot \tag{11}$$

The respective integrals (10), (11) are for prolate and oblate spheroids, and are given as (1.2.41.8) and (1.2.46.8) by [1].

1.4 $\cos^{2n}\theta$ quadrature formulas

For $\mathbf{E}_0$ polarized along the axis of rotational symmetry one finds in the case of a *prolate* spheroid

$$\int \frac{\cos^{2n}\theta}{2r}\,dV = \pi a^2 \begin{cases} \frac{1}{e}\,\mathrm{arctanh}\, e, & n=0 \\ \frac{1}{1-e^2}\,L_z, & n=1 \\ \frac{1}{3e^2}\left[\frac{3L_z}{1-e^2}-1\right], & n=2 \end{cases} \tag{12}$$

whereas, for an *oblate* spheroid,

$$\int \frac{\cos^{2n}\theta}{2r}\,dV = \pi a^2 \begin{cases} \frac{\sqrt{1-e^2}}{e}\arcsin e, & n=0 \\ (1-e^2)L_z, & n=1 \\ \frac{1-e^2}{3e^2}\left[1-3(1-e^2)L_z\right], & n=2 \end{cases} \tag{13}$$

For a sphere the above formulas reduce to

$$\int \frac{\cos^{2n}\theta}{2r}\,dV = \frac{\pi a^2}{2n+1}\cdot \tag{14}$$

1.5 Asymptotic expansions

$$\begin{aligned}\operatorname{arctanh} z &= \frac{1}{2}\left[\ln(1+z) - \ln(1-z)\right] \\ &= \sum_{l=0}^{\infty} \frac{z^{2l+1}}{2l+1} \sim z + \frac{z^3}{3} + \frac{z^5}{5} + \mathcal{O}(z^7),\end{aligned} \tag{15}$$

$$\arcsin z \sim z + \frac{z^3}{6} + \frac{3}{40}\, z^5 + \mathcal{O}(z^7), \tag{16}$$

$$\sqrt{1-z^2} \sim 1 - \frac{z^2}{2} - \frac{z^4}{8} + \mathcal{O}(z^6). \tag{17}$$

References

[1] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series*, 2nd ed (Gordon and Breach, London, 1988).