A general constraint polynomial approach

Alexander Moroz

Wave-scattering.com

wavescattering@yahoo.com

Praha, July 11, 2018

Overview

- Motivation - Kus construction for the Rabi model

Overview

- Motivation - Kus construction for the Rabi model
- General constraint polynomial approach - basic principles

Overview

- Motivation - Kus construction for the Rabi model
- General constraint polynomial approach - basic principles
- General constraint polynomial approach - application to Rabi models [JPA 51, 295201 (2018)]

Overview

- Motivation - Kus construction for the Rabi model
- General constraint polynomial approach - basic principles
- General constraint polynomial approach - application to Rabi models [JPA 51, 295201 (2018)]
- Constraint polynomial approach for quasi-exactly solvable potentials - in collaboration with Andrey Miroshnichenko

Overview

- Motivation - Kus construction for the Rabi model
- General constraint polynomial approach - basic principles
- General constraint polynomial approach - application to Rabi models [JPA 51, 295201 (2018)]
- Constraint polynomial approach for quasi-exactly solvable potentials - in collaboration with Andrey Miroshnichenko
- Conclusions

Rabi model Hamiltonian $\hat{H}_R$

$$\begin{aligned} \hat{H}_R &= \hbar\omega \mathbb{1} \hat{a}^\dagger \hat{a} + \hbar\Delta\sigma_3 + \hbar g \sigma_1 (\hat{a}^\dagger + \hat{a}) \\ &= \hbar\omega \mathbb{1} \hat{a}^\dagger \hat{a} + \hbar\Delta + \hbar g (\sigma_+ + \sigma_-)(\hat{a}^\dagger + \hat{a}) \end{aligned}$$

ω ... cavity mode frequency, $[\hat{a}, \hat{a}^\dagger] = 1$
$\mathbb{1}$ ... the unit matrix, g ... a dipole coupling constant
$\Delta = \omega_0/2$, ω_0 ... TLS resonance frequency
σ_j ... the Pauli matrices
$\kappa = g/\omega$ ... dimensionless coupling strength
$\mu = \Delta/\omega = \omega_0/(2\omega)$
Jaynes-Cummings (JC) model:

$$\hat{H}_{JC} = \hbar\omega \mathbb{1} \hat{a}^\dagger \hat{a} + \hbar\Delta\sigma_3 + \hbar g (\sigma_+ \hat{a} + \sigma_- \hat{a}^\dagger)$$

ODE for the Rabi model - [Schweber, Ann. Phys. 41, 205 (1967)]

- *Fock-Bargmann* representation with $a \to \mathrm{d}_z$ and $a^\dagger \to z$ yields for the two-component wave function $\psi(z) = \begin{pmatrix} \psi_+(z) \\ \psi_-(z) \end{pmatrix}$ after the substitution

$$\psi_\pm(z) = e^{-gz/\omega}\phi_\pm(z),$$

and on eliminating $\phi_-(z)$

$$(\omega^2 z^2 - g^2)\frac{d^2\phi_+}{dz^2} + \Big[-2\omega g z^2 + (\omega^2 - 2g^2 - 2E\omega)z + \frac{g}{\omega}(2g^2 - \omega^2)\Big]\frac{d\phi_+}{dz} + \left[2g\left(\frac{g^2}{\omega} + E\right)z + E^2 - \Delta^2 - \frac{g^4}{\omega^2}\right]\phi_+ = 0$$

ODE for the Rabi model - [Schweber, Ann. Phys. 41, 205 (1967)]

- *Fock-Bargmann* representation with $a \to \mathrm{d}_z$ and $a^\dagger \to z$ yields for the two-component wave function $\psi(z) = \begin{pmatrix} \psi_+(z) \\ \psi_-(z) \end{pmatrix}$ after the substitution

$$\psi_\pm(z) = e^{-gz/\omega}\phi_\pm(z),$$

and on eliminating $\phi_-(z)$

$$(\omega^2 z^2 - g^2)\frac{d^2\phi_+}{dz^2} + \left[-2\omega g z^2 + (\omega^2 - 2g^2 - 2E\omega)z + \frac{g}{\omega}(2g^2 - \omega^2)\right]\frac{d\phi_+}{dz} + \left[2g\left(\frac{g^2}{\omega} + E\right)z + E^2 - \Delta^2 - \frac{g^4}{\omega^2}\right]\phi_+ = 0$$

- The necessary condition for the existence of a polynomial solution of degree n is $E_n = n\omega - \frac{g^2}{\omega}$, the familiar *baseline* condition for the Rabi model.

ODE for the Rabi model - [Schweber, Ann. Phys. 41, 205 (1967)]

- *Fock-Bargmann* representation with $a \to \mathrm{d}_z$ and $a^\dagger \to z$ yields for the two-component wave function $\psi(z) = \begin{pmatrix} \psi_+(z) \\ \psi_-(z) \end{pmatrix}$ after the substitution

$$\psi_\pm(z) = e^{-gz/\omega}\phi_\pm(z),$$

and on eliminating $\phi_-(z)$

$$(\omega^2 z^2 - g^2)\frac{d^2\phi_+}{dz^2} + \left[-2\omega g z^2 + (\omega^2 - 2g^2 - 2E\omega)z + \frac{g}{\omega}(2g^2 - \omega^2)\right]\frac{d\phi_+}{dz} + \left[2g\left(\frac{g^2}{\omega} + E\right)z + E^2 - \Delta^2 - \frac{g^4}{\omega^2}\right]\phi_+ = 0$$

- The necessary condition for the existence of a polynomial solution of degree n is $E_n = n\omega - \frac{g^2}{\omega}$, the familiar *baseline* condition for the Rabi model.
- The baseline condition coincides with sl_2 algebraization

Kus construction - recapitulation

- An exact polynomial solution of the Rabi model on the *n*th baseline can be constructed as a finite linear combination of the solutions $\Phi_l^{\pm}$ of a *displaced harmonic oscillator* ($\Delta = 0$ limit of the Rabi model) from all baselines $l \leq n$ and of the same parity.

Kus construction - recapitulation

- An exact polynomial solution of the Rabi model on the nth baseline can be constructed as a finite linear combination of the solutions $\Phi_l^{\pm}$ of a *displaced harmonic oscillator* ($\Delta = 0$ limit of the Rabi model) from all baselines $l \leq n$ and of the same parity.
- For positive parity:

$$\Psi_n^+ = (2\kappa)^n \Phi_n^+ + \mu \sum_{l=1}^{n} \frac{(2\kappa)^{n-l}}{l!} K_{n,l-1} \left(\mu \Phi_{n-l}^+ + l \tilde{\Phi}_{n-l}^+ \right),$$

where Φ_l^+ is the displaced harmonic oscillator solution on the lth baseline for $\kappa = g/\omega$ and $\tilde{\Phi}_l^+$ is the solution on the lth baseline for $-g/\omega$, all solutions being of positive parity,

$$\Phi_l^+ = \begin{pmatrix} (z+\kappa)^l e^{-\kappa z} \\ (-1)^k (z-\kappa)^l e^{\kappa z} \end{pmatrix}$$

$$\tilde{\Phi}_l^+ = \begin{pmatrix} (-1)^k (z-\kappa)^l e^{\kappa z} \\ (z+\kappa)^l e^{-\kappa z} \end{pmatrix}$$

Kus recurrence - [JMP 26, 2792 (1985)]

- The polynomial K_{nn} is defined by its own *finite* three-term recurrence rescaled variables $\kappa = g/\omega$, $\mu = \Delta/\omega$ for each $n \geq 0$,

$$K_{n0} = 1, \qquad K_{n1} = 4\kappa^2 + \mu^2 - 1$$
$$K_{nl} = (4l\kappa^2 + \mu^2 - l^2)K_{n,l-1} - 4l(l-1)(n-l+1)\kappa^2 K_{n,l-2}$$

Kus recurrence - [JMP 26, 2792 (1985)]

- The polynomial K_{nn} is defined by its own *finite* three-term recurrence rescaled variables $\kappa = g/\omega$, $\mu = \Delta/\omega$ for each $n \geq 0$,

$$K_{n0} = 1, \qquad K_{n1} = 4\kappa^2 + \mu^2 - 1$$
$$K_{nl} = (4l\kappa^2 + \mu^2 - l^2)K_{n,l-1} - 4l(l-1)(n-l+1)\kappa^2 K_{n,l-2}$$

- The Kus polynomials are *not* orthogonal polynomials of independent variable κ^2, because the coefficient of $K_{n,l-2}$ depends on it!

Kus recurrence - [JMP 26, 2792 (1985)]

- The polynomial K_{nn} is defined by its own *finite* three-term recurrence rescaled variables $\kappa = g/\omega$, $\mu = \Delta/\omega$ for each $n \geq 0$,

$$K_{n0} = 1, \qquad K_{n1} = 4\kappa^2 + \mu^2 - 1$$
$$K_{nl} = (4l\kappa^2 + \mu^2 - l^2)K_{n,l-1} - 4l(l-1)(n-l+1)\kappa^2 K_{n,l-2}$$

- The Kus polynomials are *not* orthogonal polynomials of independent variable κ^2, because the coefficient of $K_{n,l-2}$ depends on it!
- In order that the Rabi model had a polynomial solution on the nth baseline, the model parameters $4\kappa^2$ and μ^2 have to be such that

$$K_{nn}(4\kappa^2, \mu^2) = 0$$

Kus recurrence - [JMP 26, 2792 (1985)]

- The polynomial K_{nn} is defined by its own *finite* three-term recurrence rescaled variables $\kappa = g/\omega$, $\mu = \Delta/\omega$ for each $n \geq 0$,

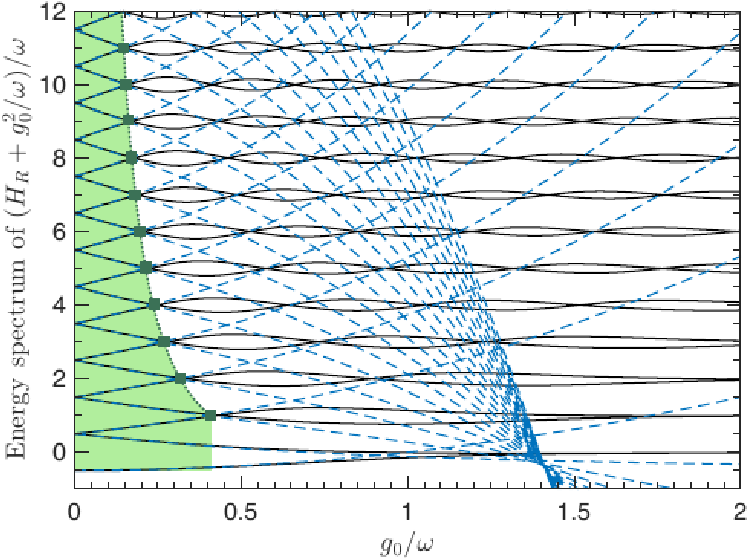
$$K_{n0} = 1, \qquad K_{n1} = 4\kappa^2 + \mu^2 - 1$$
$$K_{nl} = (4l\kappa^2 + \mu^2 - l^2)K_{n,l-1} - 4l(l-1)(n-l+1)\kappa^2 K_{n,l-2}$$

- The Kus polynomials are *not* orthogonal polynomials of independent variable κ^2, because the coefficient of $K_{n,l-2}$ depends on it!
- In order that the Rabi model had a polynomial solution on the nth baseline, the model parameters $4\kappa^2$ and μ^2 have to be such that

$$K_{nn}(4\kappa^2, \mu^2) = 0$$

- There are n simple roots for $0 < \mu < 1$

Example from Solano et al, PRA **96**, 013849 (2017)



Example from Feranchuk et al, JPA **29**, 4035 (1996)

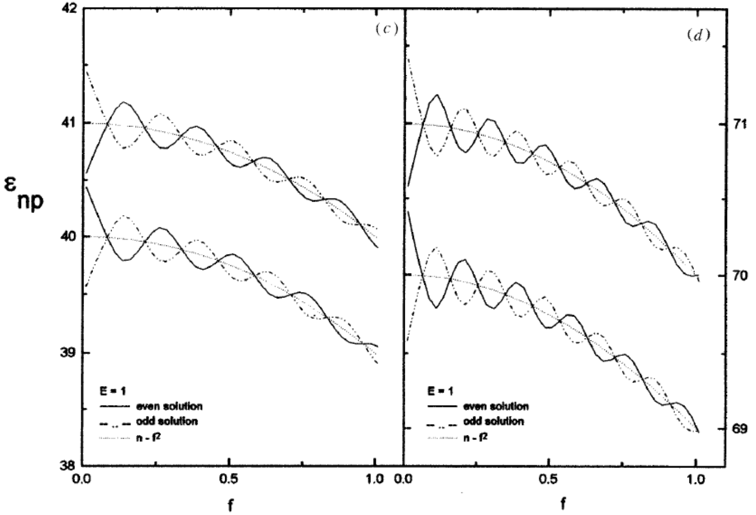


Figure 1. (*a*), (*b*) Exact and approximate eigenvalues of the TLS Hamiltonian as the functions of the coupling constant and the separation energy; (*c*), (*d*) highly excited states of the TLS.

Gradation slicing - I

- Ordinary linear differential equation (ODE) with polynomial coefficients

$$\mathcal{L}S_n(z) = \left\{A(z)\frac{d^2}{dz^2} + B(z)\frac{d}{dz} + C(z)\right\} S_n(z) = 0$$

where

$$A(z) = \sum_{k=0} a_k z^k, \quad B(z) = \sum_{k=0} b_k z^k, \quad C(z) = \sum_{k=0} c_k z^k$$

Gradation slicing - I

- Ordinary linear differential equation (ODE) with polynomial coefficients

$$\mathcal{L}S_n(z) = \left\{A(z)\frac{d^2}{dz^2} + B(z)\frac{d}{dz} + C(z)\right\} S_n(z) = 0$$

where

$$A(z) = \sum_{k=0} a_k z^k, \qquad B(z) = \sum_{k=0} b_k z^k, \qquad C(z) = \sum_{k=0} c_k z^k$$

- The *grade* of a term $z^m d_z^l$ ... the integer $m - l$

Gradation slicing - II

(S1) Consider a given differential equation as a linear combination of terms $\sim z^m d_z^l$ and determine the grade of each term.

Gradation slicing - II

(S1) Consider a given differential equation as a linear combination of terms $\sim z^m d_z^l$ and determine the grade of each term.

(S2) Rearrange all the terms of the ODE according to their grade. The subset $\mathcal{F}_{\mathfrak{g}}(z^m d_z^l; m - l \equiv \mathfrak{g})$ of the ODE with an identical grade $\mathfrak{g}$ will be called a *slice*.

Gradation slicing - II

(S1) Consider a given differential equation as a linear combination of terms $\sim z^m d_z^l$ and determine the grade of each term.

(S2) Rearrange all the terms of the ODE according to their grade. The subset $\mathcal{F}_{\mathfrak{g}}(z^m d_z^l; m - l \equiv \mathfrak{g})$ of the ODE with an identical grade $\mathfrak{g}$ will be called a *slice*.

- The differential equation can be recast as

$$\mathcal{L}S_n(z) = \sum_{\mathfrak{g}=\mathfrak{g}_{min}}^{\mathfrak{g}_{max}} \mathcal{F}_{\mathfrak{g}} S_n(z) = 0,$$

where the sum runs over all grades $\mathfrak{g}$.

Gradation slicing - II

(S1) Consider a given differential equation as a linear combination of terms $\sim z^m d_z^l$ and determine the grade of each term.

(S2) Rearrange all the terms of the ODE according to their grade. The subset $\mathcal{F}_{\mathfrak{g}}(z^m d_z^l; m - l \equiv \mathfrak{g})$ of the ODE with an identical grade $\mathfrak{g}$ will be called a *slice*.

- The differential equation can be recast as

$$\mathcal{L}S_n(z) = \sum_{\mathfrak{g}=\mathfrak{g}_{min}}^{\mathfrak{g}_{max}} \mathcal{F}_{\mathfrak{g}} S_n(z) = 0,$$

where the sum runs over all grades $\mathfrak{g}$.

- Notation: In what follows we will use an abbreviation $\gamma = \mathfrak{g}_{max}$ for the highest grade and $\gamma_* = \mathfrak{g}_{min}$ for the lowest grade.

Gradation slicing - III

- **Definition:** A decomposition of original ordinary linear differential equation into a sum of slices will be called gradation slicing.

Gradation slicing - III

- **Definition:** A decomposition of original ordinary linear differential equation into a sum of slices will be called gradation slicing.
- We call the *grade* of an ordinary linear differential equation the highest grade γ.

Gradation slicing - III

- **Definition:** A decomposition of original ordinary linear differential equation into a sum of slices will be called gradation slicing.
- We call the *grade* of an ordinary linear differential equation the highest grade γ.
- A *width* w of the gradation slicing will be called the integer $w := \gamma - \gamma_* + 1$.

Gradation slicing - III

- **Definition:** A decomposition of original ordinary linear differential equation into a sum of slices will be called gradation slicing.
- We call the *grade* of an ordinary linear differential equation the highest grade γ.
- A *width* w of the gradation slicing will be called the integer $w := \gamma - \gamma_* + 1$.
- The function $F_{\mathfrak{g}}(k)$ defined by

$$\mathcal{F}_{\mathfrak{g}} z^k := F_{\mathfrak{g}}(k) z^{k+\mathfrak{g}}.$$

will be called an *induced multiplicator* corresponding to the slice $\mathcal{F}_{\mathfrak{g}}$.

Gradation slicing - III

- **Definition:** A decomposition of original ordinary linear differential equation into a sum of slices will be called gradation slicing.
- We call the *grade* of an ordinary linear differential equation the highest grade γ.
- A *width* w of the gradation slicing will be called the integer $w := \gamma - \gamma_* + 1$.
- The function $F_{\mathfrak{g}}(k)$ defined by

$$\mathcal{F}_{\mathfrak{g}} z^k := F_{\mathfrak{g}}(k) z^{k+\mathfrak{g}}.$$

  will be called an *induced multiplicator* corresponding to the slice $\mathcal{F}_{\mathfrak{g}}$.
- The width counts the number of possible slices with the grade between the minimal and maximal grades, γ_* and γ, respectively.

Gradation slicing - III

- **Definition:** A decomposition of original ordinary linear differential equation into a sum of slices will be called gradation slicing.
- We call the *grade* of an ordinary linear differential equation the highest grade γ.
- A *width* w of the gradation slicing will be called the integer $w := \gamma - \gamma_* + 1$.
- The function $F_{\mathfrak{g}}(k)$ defined by

$$\mathcal{F}_{\mathfrak{g}} z^k := F_{\mathfrak{g}}(k) z^{k+\mathfrak{g}}.$$

  will be called an *induced multiplicator* corresponding to the slice $\mathcal{F}_{\mathfrak{g}}$.
- The width counts the number of possible slices with the grade between the minimal and maximal grades, γ_* and γ, respectively.
- Unless $C(z)$ is identically zero, one has always $\gamma \geq 0$.

Gradation slicing - III

- **Definition:** A decomposition of original ordinary linear differential equation into a sum of slices will be called gradation slicing.
- We call the *grade* of an ordinary linear differential equation the highest grade γ.
- A *width* w of the gradation slicing will be called the integer $w := \gamma - \gamma_* + 1$.
- The function $F_\mathfrak{g}(k)$ defined by

$$\mathcal{F}_\mathfrak{g} z^k := F_\mathfrak{g}(k) z^{k+\mathfrak{g}}.$$

  will be called an *induced multiplicator* corresponding to the slice $\mathcal{F}_\mathfrak{g}$.
- The width counts the number of possible slices with the grade between the minimal and maximal grades, γ_* and γ, respectively.
- Unless $C(z)$ is identically zero, one has always $\gamma \geq 0$.
- We shall assume that $\gamma_* \leq 0$. The case $\gamma_* > 0$ can always be reduced to the case $\gamma_* = 0$ by factorizing z^{γ_*} out of the polynomial coefficients of the differential equation.

Gradation slicing IV - Examples

- A hypergeometric equation is characterized by $\gamma = 0$, $\gamma_* = -2$, and $w = 3$.

Gradation slicing IV - Examples

- A hypergeometric equation is characterized by $\gamma = 0$, $\gamma_* = -2$, and $w = 3$.
- A typical Heine-Stieltjes problem, where $A(z), B(z), C(z)$ are *polynomials* of exact degree $N+2$, $N+1$, N, respectively, is grade $\gamma = N$, $\gamma_* = -2$, $w = N+3$ problem.

Gradation slicing IV - Examples

- A hypergeometric equation is characterized by $\gamma = 0$, $\gamma_* = -2$, and $w = 3$.
- A typical Heine-Stieltjes problem, where $A(z), B(z), C(z)$ are *polynomials* of exact degree $N+2$, $N+1$, N, respectively, is grade $\gamma = N$, $\gamma_* = -2$, $w = N+3$ problem.
- The condition that the slice with the highest grade γ annihilates a monomial of degree n, $\mathcal{F}_\gamma z^n = 0$, provides a *necessary* condition for the existence of a polynomial solution of degree n,

$$F_\gamma(n) = 0$$

Gradation slicing V - Examples - Two main alternative types of differential equations

(A1) A *conventional* one when the highest derivative term [e.g. $A(z)d_z^2$ in ODE does contribute to the slice $\mathcal{F}_\gamma$ with the highest grade γ. The alternative occurs if

$$\deg B \leq \deg A - 1, \qquad \deg C \leq \deg A - 2$$

where strict equality applies in at least one of the above cases.

Gradation slicing V - Examples - Two main alternative types of differential equations

(A1) A *conventional* one when the highest derivative term [e.g. $A(z)d_z^2$ in ODE does contribute to the slice $\mathcal{F}_\gamma$ with the highest grade γ. The alternative occurs if

$$\deg B \le \deg A - 1, \qquad \deg C \le \deg A - 2$$

where strict equality applies in at least one of the above cases.

- Examples are the Fuchsian equations, which include a hypergeometric one, and the Heine-Stieltjes problem.

Gradation slicing V - Examples - Two main alternative types of differential equations

(A1) A *conventional* one when the highest derivative term [e.g. $A(z)d_z^2$ in ODE does contribute to the slice $\mathcal{F}_\gamma$ with the highest grade γ. The alternative occurs if

$$\deg B \leq \deg A - 1, \qquad \deg C \leq \deg A - 2$$

where strict equality applies in at least one of the above cases.

- Examples are the Fuchsian equations, which include a hypergeometric one, and the Heine-Stieltjes problem.

(A2) An *anomalous* one, when the highest derivative term [e.g. $A(z)d_z^2$ in ODE does *not* contribute to $\mathcal{F}_\gamma$. Applied to ODE, the alternative occurs if

$$\deg B \geq \deg A, \qquad \deg C = \deg B - 1$$

Gradation slicing V - Examples - Two main alternative types of differential equations

(A1) A *conventional* one when the highest derivative term [e.g. $A(z)d_z^2$ in ODE does contribute to the slice $\mathcal{F}_\gamma$ with the highest grade γ. The alternative occurs if

$$\deg B \leq \deg A - 1, \qquad \deg C \leq \deg A - 2$$

where strict equality applies in at least one of the above cases.

- Examples are the Fuchsian equations, which include a hypergeometric one, and the Heine-Stieltjes problem.

(A2) An *anomalous* one, when the highest derivative term [e.g. $A(z)d_z^2$ in ODE does *not* contribute to $\mathcal{F}_\gamma$. Applied to ODE, the alternative occurs if

$$\deg B \geq \deg A, \qquad \deg C = \deg B - 1$$

- Examples: all Rabi model examples considered.

General theory

- $$S_n(z) = \prod_{i=1}^{n}(z - z_i) = \sum_{k=0}^{n} a_{nk} z^k \qquad\qquad (a_{nn} \equiv 1)$$

General theory

- $$S_n(z) = \prod_{i=1}^{n}(z - z_i) = \sum_{k=0}^{n} a_{nk} z^k \qquad (a_{nn} \equiv 1)$$

- $$\begin{aligned}
F_{\gamma-1}(n) + a_{n,n-1}F_\gamma(n-1) &= 0\\
F_{\gamma-2}(n) + a_{n,n-1}F_{\gamma-1}(n-1) + a_{n,n-2}F_\gamma(n-2) &= 0\\
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad &\vdots\\
a_{n,\gamma+2-\gamma_*}F_{\gamma_*}(\gamma+2-\gamma_*) + \ldots + a_{n3}F_{\gamma-1}(3) + a_{n2}F_\gamma(2) &= 0\\
a_{nw}F_{\gamma_*}(w) + \ldots + a_{n2}F_{\gamma-1}(2) + a_{n1}F_\gamma(1) &= 0\\
a_{n,w-1}F_{\gamma_*}(w-1) + \ldots + a_{n1}F_{\gamma-1}(1) + a_{n0}F_\gamma(0) &= 0
\end{aligned}$$

(We recall that $w = \gamma - \gamma_* + 1$ is the gradation slicing width.)

General theory

- $$S_n(z) = \prod_{i=1}^{n}(z - z_i) = \sum_{k=0}^{n} a_{nk} z^k \qquad (a_{nn} \equiv 1)$$

- $$\begin{array}{r} F_{\gamma-1}(n) + a_{n,n-1}F_\gamma(n-1) = 0 \\ F_{\gamma-2}(n) + a_{n,n-1}F_{\gamma-1}(n-1) + a_{n,n-2}F_\gamma(n-2) = 0 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{n,\gamma+2-\gamma_*}F_{\gamma_*}(\gamma+2-\gamma_*) + \ldots + a_{n3}F_{\gamma-1}(3) + a_{n2}F_\gamma(2) = 0 \\ a_{nw}F_{\gamma_*}(w) + \ldots + a_{n2}F_{\gamma-1}(2) + a_{n1}F_\gamma(1) = 0 \\ a_{n,w-1}F_{\gamma_*}(w-1) + \ldots + a_{n1}F_{\gamma-1}(1) + a_{n0}F_\gamma(0) = 0 \end{array}$$

  (We recall that $w = \gamma - \gamma_* + 1$ is the gradation slicing width.)
- Each line summarizes all the terms contributing to the same power of z, beginning from $n - 1 + \gamma$ of the first equation down to γ of the last equation.

Basic theorem for $\gamma = 0$

- **Theorem 1:** A necessary and sufficient conditions for the ODE with the grade $\gamma = 0$ to have a *unique* polynomial solution of nth degree is that
$$F_\gamma(n) = 0, \qquad F_\gamma(k) \neq 0 \quad (0 \leq k < n)$$
where the second condition applies for $n \geq 1$. $\square$

Basic theorem for $\gamma = 0$

- **Theorem 1:** A necessary and sufficient conditions for the ODE with the grade $\gamma = 0$ to have a *unique* polynomial solution of nth degree is that
$$F_\gamma(n) = 0, \qquad F_\gamma(k) \neq 0 \quad (0 \leq k < n)$$
where the second condition applies for $n \geq 1$. $\square$
- The condition $F_0(n) = 0$ is obviously necessary.

Basic theorem for $\gamma = 0$

- **Theorem 1:** A necessary and sufficient conditions for the ODE with the grade $\gamma = 0$ to have a *unique* polynomial solution of nth degree is that

$$F_\gamma(n) = 0, \qquad F_\gamma(k) \neq 0 \quad (0 \leq k < n)$$

where the second condition applies for $n \geq 1$. $\square$

- The condition $F_0(n) = 0$ is obviously necessary.
- In order to demonstrate sufficiency, note that the second condition $F_0(k) \neq 0$ ensures that each subsequent line in the system $(\star)$ of n equations, when progressing from the very top down, enables one to *uniquely* determine newly appearing coefficient (i.e. $a_{n,n-l}$ in the lth line) and thus to determine at the end a unique set of coefficients a_{nk}, $0 \leq k < n$. The initial condition $a_{nn} = 1$ is used here to simply fix an arbitrary irrelevant multiplication factor. The point of crucial importance is that for (and only for) $\gamma = 0$ the image $\mathcal{L}S_n(z)$ and $S_n(z)$ are polynomials of the same degree.

Basic theorem for $\gamma = 0$ - JPA 51, 295201 (2018)

- **Corollary 2:** If $F_\gamma(k)$ is a *linear* function of k, there is always at most a single unique polynomial solution, because a linear function can have at most a single root. $\square$

- **Theorem 2:** A necessary and sufficient conditions for the ODE with the grade $\gamma > 0$ to have a unique polynomial solution is that, in addition to the conditions of Theorem 1 which determine the unique set of coefficients $\{a_{nk}\}_{k=0}^{n}$ by the recurrence $(\star)$, the subset $\{a_{n0}, a_{n1}, \ldots, a_{n,w-2}\}$ of the coefficients $\{a_{nk}\}_{k=0}^{n}$ satisfies additional γ constraints $(\star\star)$:

$$
\begin{aligned}
\mathcal{P}_\gamma &:= a_{n,w-2}F_{\gamma_*}(w-2) + \ldots + a_{n1}F_{\gamma-2}(1) + a_{n0}F_{\gamma-1}(0) = 0 \\
&\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
\mathcal{P}_2 &:= a_{n,1-\gamma_*}F_{\gamma_*}(1-\gamma_*) + \ldots + a_{n1}F_0(1) + a_{n0}F_1(0) = 0 \\
\mathcal{P}_1 &:= a_{n,-\gamma_*}F_{\gamma_*}(-\gamma_*) + \ldots + a_{n1}F_{-1}(1) + a_{n0}F_0(0) = 0
\end{aligned}
$$

Basic theorem for $\gamma \neq 0$ - II

Proof: According to the definition, $\mathcal{F}_\gamma z^0 \sim z^\gamma$. Therefore, whenever $\gamma > 0$, the recurrence $(\star)$ does not take into account the terms $\sim z^k$ of degree $k < \gamma$ of the image $\mathcal{L}S_n(z)$. There are exactly γ of such polynomial terms with $k = 0, \ldots, \gamma - 1$. One can verify that the vanishing of the coefficients of z^k, $0 \leq k < \gamma$ amounts to solving the system $(\star)$. The vanishing of the coefficients of z^k, $k < \gamma$ thus imposes γ constraints on the (up to a multiplication by a constant) unique set of coefficients a_{nk}.

Basic theorem for $\gamma \neq 0$ - II

Proof: According to the definition, $\mathcal{F}_\gamma z^0 \sim z^\gamma$. Therefore, whenever $\gamma > 0$, the recurrence $(\star)$ does not take into account the terms $\sim z^k$ of degree $k < \gamma$ of the image $\mathcal{L}S_n(z)$. There are exactly γ of such polynomial terms with $k = 0, \ldots, \gamma - 1$. One can verify that the vanishing of the coefficients of z^k, $0 \leq k < \gamma$ amounts to solving the system $(\star)$. The vanishing of the coefficients of z^k, $k < \gamma$ thus imposes γ constraints on the (up to a multiplication by a constant) unique set of coefficients a_{nk}.

Theorem 3: Provided that each $F_{\mathfrak{g}}(k)$ is a polynomial in model parameter(s) and the hypotheses of Theorem 2 are satisfied, each recursively determined $P_{\mathfrak{g}}$, $\mathfrak{g} = 1, \ldots, \gamma$ of Eq. $(\star)$ is proportional to a polynomial in model parameter(s). $\square$

An application to the Rabi model

- We can read polynomial coefficients a_j, b_j, c_j from

$$(\omega^2 z^2 - g^2)\frac{d^2\phi_+}{dz^2} + \Big[-2\omega g z^2 + (\omega^2 - 2g^2 - 2E\omega)z +$$

$$\frac{g}{\omega}(2g^2 - \omega^2)\Big]\frac{d\phi_+}{dz} + \left[2g\left(\frac{g^2}{\omega} + E\right)z + E^2 - \Delta^2 - \frac{g^4}{\omega^2}\right]\phi_+ = 0$$

An application to the Rabi model

- We can read polynomial coefficients a_j, b_j, c_j from

$$(\omega^2 z^2 - g^2)\frac{d^2\phi_+}{dz^2} + \Big[-2\omega g z^2 + (\omega^2 - 2g^2 - 2E\omega)z + \frac{g}{\omega}(2g^2 - \omega^2)\Big]\frac{d\phi_+}{dz} + \left[2g\left(\frac{g^2}{\omega} + E\right)z + E^2 - \Delta^2 - \frac{g^4}{\omega^2}\right]\phi_+ = 0$$

- Focusing on the nth baseline, $E_n = n\omega - (g^2/\omega)$, which is the solution of $F_1(n) = 0$, one finds $b_1(E_n) = (1-2n)\omega^2$, $c_0(E_n) = n^2\omega^2 - \Delta^2 - 2ng^2$, and

$$F_1(k) = 2\omega g(n-k)$$
$$F_0(k) = k(k-2n)\omega^2 + n^2\omega^2 - \Delta^2 - 2ng^2$$
$$F_{-1}(k) = \tfrac{kg}{\omega}(2g^2 - \omega^2), \qquad F_{-2}(k) = -k(k-1)g^2$$

(The baseline condition coincides with sl_2 algebraization!)

- We can read polynomial coefficients a_j, b_j, c_j from

$$(\omega^2 z^2 - g^2)\frac{d^2\phi_+}{dz^2} + \Big[-2\omega g z^2 + (\omega^2 - 2g^2 - 2E\omega)z + \frac{g}{\omega}(2g^2 - \omega^2)\Big]\frac{d\phi_+}{dz} + \left[2g\left(\frac{g^2}{\omega} + E\right)z + E^2 - \Delta^2 - \frac{g^4}{\omega^2}\right]\phi_+ = 0$$

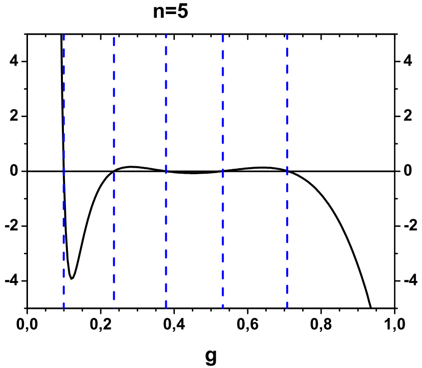
- Focusing on the nth baseline, $E_n = n\omega - (g^2/\omega)$, which is the solution of $F_1(n) = 0$, one finds $b_1(E_n) = (1-2n)\omega^2$, $c_0(E_n) = n^2\omega^2 - \Delta^2 - 2ng^2$, and

$$F_1(k) = 2\omega g(n-k)$$
$$F_0(k) = k(k-2n)\omega^2 + n^2\omega^2 - \Delta^2 - 2ng^2$$
$$F_{-1}(k) = \tfrac{kg}{\omega}(2g^2-\omega^2), \qquad F_{-2}(k) = -k(k-1)g^2$$

(The baseline condition coincides with sl_2 algebraization!)

- $\gamma = 1 \Longrightarrow$ single polynomial constraint

A comparison of the polynomial constraint relation and the Kus polynomial for the Rabi model at the fifth baseline



The polynomials are plotted as a function of g with fixed $\Delta = 0.1$ and $\omega = 0.4$.

Rabi model generalizations

- *Driven* Rabi model

$$\hat{H}_R = \omega \mathbb{1} a^\dagger a + \Delta \sigma_z + g\, \sigma_x \left(a^\dagger + a \right) + \delta\, \sigma_x$$

Rabi model generalizations

- *Driven* Rabi model

$$\hat{H}_R = \omega \mathbb{1} a^\dagger a + \Delta \sigma_z + g\, \sigma_x \left(a^\dagger + a \right) + \delta\, \sigma_x$$

- *Two-photon* Rabi model

$$\hat{H}_{2p} = \omega \mathbb{1} a^\dagger a + \Delta \sigma_z + g\, \sigma_x \left[(a^\dagger)^2 + a^2 \right]$$

Rabi model generalizations

- *Driven* Rabi model

$$\hat{H}_R = \omega \mathbb{1} a^\dagger a + \Delta \sigma_z + g\,\sigma_x \left(a^\dagger + a\right) + \delta\,\sigma_x$$

- *Two-photon* Rabi model

$$\hat{H}_{2p} = \omega \mathbb{1} a^\dagger a + \Delta \sigma_z + g\,\sigma_x \left[(a^\dagger)^2 + a^2\right]$$

- The nonlinear *two-mode* quantum Rabi model

$$\hat{H}_{2m} = \omega \mathbb{1}(a_1^\dagger a_1 + a_2^\dagger a_2) + \Delta \sigma_z + g\,\sigma_x (a_1^\dagger a_2^\dagger + a_1 a_2)$$

Rabi model generalizations

- *Driven* Rabi model

$$\hat{H}_R = \omega \mathbb{1} a^\dagger a + \Delta \sigma_z + g\, \sigma_x \left(a^\dagger + a \right) + \delta\, \sigma_x$$

- *Two-photon* Rabi model

$$\hat{H}_{2p} = \omega \mathbb{1} a^\dagger a + \Delta \sigma_z + g\, \sigma_x \left[(a^\dagger)^2 + a^2 \right]$$

- The nonlinear *two-mode* quantum Rabi model

$$\hat{H}_{2m} = \omega \mathbb{1} (a_1^\dagger a_1 + a_2^\dagger a_2) + \Delta \sigma_z + g\, \sigma_x (a_1^\dagger a_2^\dagger + a_1 a_2)$$

- The *generalized*, or also known as an asymmetric, Rabi model

$$\hat{H}_{\mathrm{gR}} = \omega \mathbb{1} a^\dagger a + \Delta \sigma_z + g_1 \left(a^\dagger \sigma_- + a \sigma_+ \right) + g_2 \left(a^\dagger \sigma_+ + a \sigma_- \right)$$

interpolating between the Jaynes-Cummings (JC) model (for $g_2 = 0$) and the original Rabi model ($g_1 = g_2$)

An application to the driven Rabi model

- We can read polynomial coefficients a_j, b_j, c_j from

$$\hat{H}_R = (\omega^2 z^2 - g^2)\frac{d^2}{dz^2} + \Big[-2\omega g z^2 + (\omega^2 - 2g^2 - 2E\omega)z - g\omega + 2g\left(\frac{g^2}{\omega} \mp \delta\right)\Big]\frac{d}{dz} + 2g\left(\frac{g^2}{\omega} + E \mp \delta\right) z + E^2 - \left(\frac{g^2}{\omega} \mp \delta\right)^2$$

depending on the substitution $e^{-gz/\omega}\phi_\pm(z)$ or $e^{+gz/\omega}\phi_\pm(z)$

An application to the driven Rabi model

- We can read polynomial coefficients a_j, b_j, c_j from

$$\hat{H}_R = (\omega^2 z^2 - g^2)\frac{d^2}{dz^2} + \Bigg[-2\omega g z^2 + (\omega^2 - 2g^2 - 2E\omega)z - g\omega$$
$$+2g\left(\frac{g^2}{\omega} \mp \delta\right)\Bigg]\frac{d}{dz} + 2g\left(\frac{g^2}{\omega} + E \mp \delta\right)z + E^2 - \left(\frac{g^2}{\omega} \mp \delta\right)^2$$

depending on the substitution $e^{-gz/\omega}\phi_\pm(z)$ or $e^{+gz/\omega}\phi_\pm(z)$

- On the nth baseline, $E_n = n\omega - (g^2/\omega) \pm \delta$,

$$F_1(k) = \pm 2\omega g(n-k)$$
$$F_0(k) = k(k-2n)\omega^2 + n^2\omega^2 - \Delta^2 - 2ng^2 \pm 2(n-k)\omega\delta$$
$$F_{-1}(k) = \pm\tfrac{kg}{\omega}(2g^2 - \omega^2 \mp 2\omega\delta), \qquad F_{-2}(k) = -k(k-1)g^2$$

(The baseline condition coincides again with sl_2 algebraization!)

An application to the driven Rabi model

- We can read polynomial coefficients a_j, b_j, c_j from

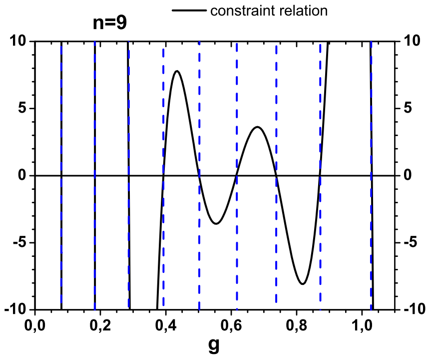
$$\hat{H}_R = (\omega^2 z^2 - g^2)\frac{d^2}{dz^2} + \Bigg[-2\omega g z^2 + (\omega^2 - 2g^2 - 2E\omega)z - g\omega + 2g\left(\frac{g^2}{\omega} \mp \delta\right)\Bigg]\frac{d}{dz} + 2g\left(\frac{g^2}{\omega} + E \mp \delta\right) z + E^2 - \left(\frac{g^2}{\omega} \mp \delta\right)^2$$

  depending on the substitution $e^{-gz/\omega}\phi_\pm(z)$ or $e^{+gz/\omega}\phi_\pm(z)$
- On the nth baseline, $E_n = n\omega - (g^2/\omega) \pm \delta$,

$$F_1(k) = \pm 2\omega g(n-k)$$
$$F_0(k) = k(k-2n)\omega^2 + n^2\omega^2 - \Delta^2 - 2ng^2 \pm 2(n-k)\omega\delta$$
$$F_{-1}(k) = \pm\frac{kg}{\omega}(2g^2 - \omega^2 \mp 2\omega\delta), \qquad F_{-2}(k) = -k(k-1)g^2$$

  (The baseline condition coincides again with sl_2 algebraization!)
- $\gamma = 1 \Longrightarrow$ single polynomial constraint

A comparison on the ninth baseline



The polynomials are plotted as a function of g with fixed $\Delta = 0.1$, $\omega = 0.4$, and $\delta = 0.02$. Dashed lines showing the generalized Kus constraint polynomial for the driven Rabi model generated by the recursion (B.1) of Z.-M. Li et al, JPA48, 454005 (2015)

$$\begin{aligned} P_0 &= 1, \quad P_1 = 4g^2 + \Delta^2 - \omega^2 - 2\delta\,\omega \\ P_k &= \left[k(2g)^2 + \Delta^2 - k^2\omega^2 - 2k\delta\,\omega\right] P_{k-1} \\ &\quad - k(k-1)(n-k+1)(2g)^2\omega^2 P_{k-2} \end{aligned}$$

Quasi-exactly solvable potentials of the Schrödinger equation - I

- The Schrödinger equation ($\hbar = 2m = 1$)

$$\left(-\frac{d^2}{dx^2} + V\right)\psi = E\psi$$

for a number of quasi-exactly solvable potentials V can on using a suitable substitution be recast in the same basic form as

$$(a_3 z^3 + a_2 z^2 + a_1 z)\frac{d^2\phi(z)}{dz^2} + (b_2 z^2 + b_1 z + b_0)\frac{d\phi(z)}{dz} + (c_1 z + c_0)\phi(z) = 0$$

where $a_3, a_2, a_1, b_2, b_1, b_0, c_1, c_0$ are constant parameters.

Quasi-exactly solvable potentials of the Schrödinger equation - I

- The Schrödinger equation ($\hbar = 2m = 1$)

$$\left(-\frac{d^2}{dx^2} + V\right)\psi = E\psi$$

for a number of quasi-exactly solvable potentials V can on using a suitable substitution be recast in the same basic form as

$$(a_3 z^3 + a_2 z^2 + a_1 z)\frac{d^2\phi(z)}{dz^2} + (b_2 z^2 + b_1 z + b_0)\frac{d\phi(z)}{dz} + (c_1 z + c_0)\phi(z) = 0$$

where $a_3, a_2, a_1, b_2, b_1, b_0, c_1, c_0$ are constant parameters.

- Three slices with the respective multiplicators

$$F_1(n) = n(n-1)a_3 + nb_2 + c_1, \qquad F_0(n) = n(n-1)a_2 + nb_1 + c_0,$$
$$F_{-1}(n) = n(n-1)a_1 + nb_0$$

QES potentials - III

$\gamma = 1 \Longrightarrow$ single constraint polynomial $(\star\star)$

$$\mathcal{P}(n) := b_0 P_{n,n-1} + c_0 P_{nn} = 0$$

QES potentials - III

$\gamma = 1 \Longrightarrow$ single constraint polynomial $(\star\star)$

$$\mathcal{P}(n) := b_0 P_{n,n-1} + c_0 P_{nn} = 0$$

Examples: two different sets of modified Manning potentials with three parameters, an electron in Coulomb and magnetic fields and relative motion of two electrons in an external oscillator potential, the hyperbolic Razavy potential, a (perturbed) double sinh-Gordon system, and many others.

QES potentials - IV

- The conditions enable one to determine unique set of coefficients $\{P_{nk}\}_{k=0}^{n}$, defined recursively by the *three-term* recurrence for $1 \leq k \leq n$ beginning with $P_{n0} = 1$ (cf. Eq. (11) of AM, JPA 51, 295201 (2018))

$$
\begin{aligned}
P_{n1} &= -F_0(n)P_{n0}/F_1(n-1) \\
P_{n2} &= -[F_{-1}(n)P_{n0} + F_0(n-1)P_{n1}]/F_1(n-2) \\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
P_{n,k} &= -[F_{-1}(n+2-k)P_{n,k-2} + F_0(n+1-k)P_{n,k-1}]/F_1(n-k) \\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
P_{nn} &= -[F_{-1}(2)P_{n,n-2} + F_0(1)P_{n,n-1}]/F_1(0)
\end{aligned}
$$

QES potentials - IV

- The conditions enable one to determine unique set of coefficients $\{P_{nk}\}_{k=0}^{n}$, defined recursively by the *three-term* recurrence for $1 \leq k \leq n$ beginning with $P_{n0} = 1$ (cf. Eq. (11) of AM, JPA 51, 295201 (2018))

$$
\begin{aligned}
P_{n1} &= -F_0(n)P_{n0}/F_1(n-1) \\
P_{n2} &= -[F_{-1}(n)P_{n0} + F_0(n-1)P_{n1}]/F_1(n-2) \\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
P_{n,k} &= -[F_{-1}(n+2-k)P_{n,k-2} + F_0(n+1-k)P_{n,k-1}]/F_1(n-k) \\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
P_{nn} &= -[F_{-1}(2)P_{n,n-2} + F_0(1)P_{n,n-1}]/F_1(0)
\end{aligned}
$$

- If the unique (monic) polynomial solution exists, then it is necessarily given by $S_n(z) = \sum_{k=0}^{n} P_{n,n-k} z^k$.

(A1) $F_1(n)$ does depend on energy $\Longrightarrow$ energy can be then expressed as a function of model parameters, $E = E(V_j)$, and thereby eliminated from recurrence coefficients and from the constraint polynomial $\mathcal{P}(n)$ by imposing the constraint $F_1(n) = 0$. $\mathcal{P}(n) = 0$ determines a *discrete* set of parameters on the nth baseline at which polynomial solutions exist, and in turn allowed energies by parametric dependence $E = E(V_j) \Longrightarrow$ entirely analogous to the Kus polynomials in the Rabi model.

QES potentials - V

(A1) $F_1(n)$ does depend on energy $\Longrightarrow$ energy can be then expressed as a function of model parameters, $E = E(V_j)$, and thereby eliminated from recurrence coefficients and from the constraint polynomial $\mathcal{P}(n)$ by imposing the constraint $F_1(n) = 0$. $\mathcal{P}(n) = 0$ determines a *discrete* set of parameters on the nth baseline at which polynomial solutions exist, and in turn allowed energies by parametric dependence $E = E(V_j) \Longrightarrow$ entirely analogous to the Kus polynomials in the Rabi model.

(A2) Only the multiplicator $F_0(k)$ depends on energy, and is a *linear* function of it $\Longrightarrow \mathcal{P}(n)$ is necessarily a polynomial of degree $n+1$ in energy and provides a kind of energy quantization rule. The latter sounds similar to the role played by a critical polynomial of the Lanczos-Haydock finite-chain of polynomials (also known as the Bender-Dunne polynomials).

QES potentials - VI

- For the QES examples considered, constraint polynomials terminate a finite orthogonal polynomial system characterized by a positive-definite moment functional $\mathcal{L}$, implying that a corresponding constraint polynomial has only real and simple zeros.

QES potentials - VI

- For the QES examples considered, constraint polynomials terminate a finite orthogonal polynomial system characterized by a positive-definite moment functional $\mathcal{L}$, implying that a corresponding constraint polynomial has only real and simple zeros.
- Any resemblance with the Bender-Dunne polynomials for the alternative (**A2**) is only coincidental, because the finite orthogonal polynomial system terminated by the constraint polynomials is shown to be different from the weak orthogonal Bender-Dunne polynomials.

QES examples - Alternative (**A1**)

- $$V(x) = -V_1 \operatorname{sech}^6 x - V_2 \operatorname{sech}^4 x - V_3 \operatorname{sech}^2 x$$

  [Xie, JPA 45, 175302 (2012)] [$V_1 \equiv 0$ ... Manning, J. Chem. Phys. 3, 136 (1935)]

QES examples - Alternative (**A1**)

- $$V(x) = -V_1 \operatorname{sech}^6 x - V_2 \operatorname{sech}^4 x - V_3 \operatorname{sech}^2 x$$

  [Xie, JPA 45, 175302 (2012)] [$V_1 \equiv 0 \ldots$ Manning, J. Chem. Phys. 3, 136 (1935)]

- $$V(x) = \frac{V_1}{\cosh^2 x} + \frac{V_2}{1 + g\cosh^2 x} + \frac{V_3}{(1 + g\cosh^2 x)^2}$$

  [Chen et al JPA 46, 035301 (2013)]

QES examples - Alternative (**A1**)

- $$V(x) = -V_1 \operatorname{sech}^6 x - V_2 \operatorname{sech}^4 x - V_3 \operatorname{sech}^2 x$$

  [Xie, JPA 45, 175302 (2012)] [$V_1 \equiv 0$ ... Manning, J. Chem. Phys. 3, 136 (1935)]

- $$V(x) = \frac{V_1}{\cosh^2 x} + \frac{V_2}{1 + g\cosh^2 x} + \frac{V_3}{(1 + g\cosh^2 x)^2}$$

  [Chen et al JPA 46, 035301 (2013)]

- $$\left[\frac{1}{2}\frac{d^2}{dr^2} - \frac{\lambda(\lambda-1)}{2}\frac{1}{r^2} - \frac{1}{2}\omega^2 r^2 + \frac{\beta}{r} + \alpha\right] u(r) = 0$$

  with real parameters β, and λ, $\omega > 0$, and eigenvalue α [Chiang et al, PRA 63, 062105 (2001)]

Chen et al 3-parameter modified Manning

- $$V(x) = \frac{V_1}{\cosh^2 x} + \frac{V_2}{1 + g\cosh^2 x} + \frac{V_3}{(1 + g\cosh^2 x)^2}$$

Chen et al 3-parameter modified Manning

- $$V(x) = \frac{V_1}{\cosh^2 x} + \frac{V_2}{1 + g\cosh^2 x} + \frac{V_3}{(1 + g\cosh^2 x)^2}$$
- The change in variable through $z = -\sinh^2 x$ and the substitution
$$\psi(x) = (\cosh x)^{2\lambda_1}(1 + g\cosh^2 x)^{\lambda_2}\sinh x\,\phi(z),$$
$$\lambda_1 = \tfrac{1}{4}\left(1 + \sqrt{1 - 4V_1}\right), \qquad \lambda_2 = \tfrac{1}{2}\left[1 - \sqrt{1 + V_3/(1+g)}\right]$$
transform the Schrödinger equation into
$$a_3 = 1, \qquad a_2 = -2 - 1/g, \qquad a_1 = 1 + 1/g$$
$$b_2 = 2(\lambda_1 + \lambda_2 + 1)$$
$$b_1 = -\left[2\lambda_1 + 2\lambda_2 + \tfrac{7}{2} + \tfrac{2(\lambda_1+1)}{g}\right]$$
$$b_0 = \tfrac{3(1+g)}{2g}, \qquad c_1 = (\lambda_1 + \lambda_2)(\lambda_1 + \lambda_2 + 1) + \tfrac{E+1}{4}$$
$$c_0 = -\tfrac{1+g}{4g}\left[6\lambda_1 + 4\lambda_2 + 1 + \tfrac{2\lambda_2 g - V_2}{1+g} - V_1 - \tfrac{V_3}{(1+g)^2} + E\right] + \tfrac{\lambda_2}{g}$$

Chen et al 3-parameter modified Manning - II

- On the nth baseline

$$E_n = -1 - 4(n + \lambda_1 + \lambda_2)(n + \lambda_1 + \lambda_2 + 1)$$

$$F_1(k) = k(k-1) - n(n-1) + 2(k-n)(\lambda_1 + \lambda_2 + 1)$$
$$F_0(k) = -k(k-1)\left(2 + \frac{1}{g}\right) - k\left[2\lambda_2 + 2\lambda_1 + \frac{7}{2} + \frac{2(\lambda_1+1)}{g}\right] + c_0(n)$$
$$F_{-1}(k) = k(k-1)\left(1 + \frac{1}{g}\right) + \frac{3}{2}k\left(1 + \frac{1}{g}\right) = \frac{1+g}{2g}k(2k+1)$$

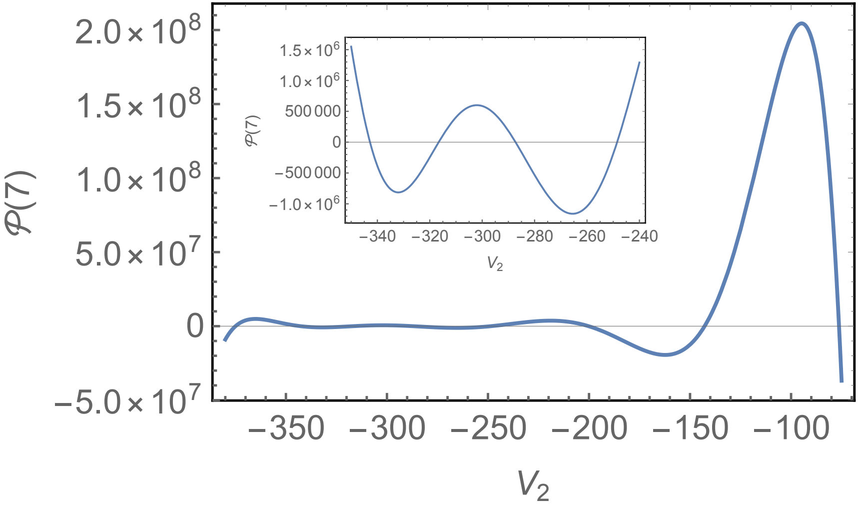
where

$$c_0(n) = -\frac{1+g}{4g}\left\{6\lambda_1 + 4\lambda_2 + 1 + \frac{2\lambda_2 g - V_2}{1+g} - V_1 - \frac{V_3}{(1+g)^2} - 1 - 4(n + \lambda_1 + \lambda_2)(n + \lambda_1 + \lambda_2 + 1)\right\} + \frac{\lambda_2}{g}$$

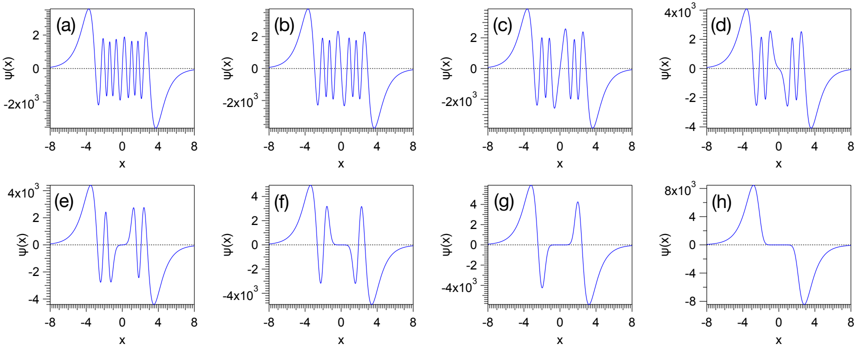
Chen et al 3-parameter modified Manning - III

$$
\begin{aligned}
P_{n1} &= -F_0(n)P_{n0}/F_1(n-1) \\
\vdots \qquad & \qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
P_{n,k} &= -[F_0(n+1-k)/F_1(n-k)]P_{n,k-1} - [F_{-1}(n+2-k)/F_1(n-k \\
\vdots \qquad & \qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
P_{nn} &= -[F_0(1)/F_1(0)]P_{n,n-1}] - [F_{-1}(2)/F_1(0)]P_{n,n-2} \\
\mathcal{P}(n) &= F_{-1}(1)P_{n,n-1} + F_0(1)P_{nn} = b_0 P_{n,n-1} + c_0 P_{nn}
\end{aligned}
$$

defines for nth baseline a finite orthogonal polynomial system $\{P_{nk},\ k = 0, 1, 2, \ldots, n,\ \mathcal{P}(n)\}$ in $V_2 \Longrightarrow$ only simple roots



Constraint polynomial for the Chen et al. generalized Manning potential in the *odd* parity case as a function of V_2 with fixed $V_1 = 0.09$, $V_3 = 400$, $g = 0.25$, and $n = 7$. There is the maximum number of 8 *real* zeros: $V_2 = -374.929, -342.812, -316.597, -287.269, -248.489, -200.236, -142.792, -76.2691$.



Polynomial eigenfunctions for the Chen et al. generalized Manning potential in the *odd* parity case with fixed $V_1 = 1$, $V_3 = 400$, $g = 0.25$, and $n = 7$ for the values of
$V_2 = -374.929,\ -342.812,\ -316.597,\ -287.269,\ -248.489,\ -200.236,\ -142.792,\ -76.2691.$

QES examples - Alternative (**A2**)

- The hyperbolic Razavy potential [Am. J. Phys. 48, 285 (1980)]

$$V(x) = \frac{1}{4}\xi^2 \sinh^2(2x) - (N+1)\xi \cosh(2x)$$

QES examples - Alternative (**A2**)

- The hyperbolic Razavy potential [Am. J. Phys. 48, 285 (1980)]

$$V(x) = \frac{1}{4}\xi^2 \sinh^2(2x) - (N+1)\xi \cosh(2x)$$

- The double sinh-Gordon (DSHG) parity invariant system (also called the bistable Razavy potential)

$$V(x) = [\xi \cosh(2x) - M]^2$$

(Baradaran et al, arXiv:1712.06439)

QES examples - Alternative (**A2**)

- The hyperbolic Razavy potential [Am. J. Phys. 48, 285 (1980)]

$$V(x) = \frac{1}{4}\xi^2 \sinh^2(2x) - (N+1)\xi \cosh(2x)$$

- The double sinh-Gordon (DSHG) parity invariant system (also called the bistable Razavy potential)

$$V(x) = [\xi \cosh(2x) - M]^2$$

  (Baradaran et al, arXiv:1712.06439)

- DSHG perturbed by

$$V_p = -\frac{g(g+1)}{\cosh^2 x} + \frac{h(h+1)}{\sinh^2 x}$$

  [Khare et al, Pramana J. Phys. 73, 387 (2009)]

The double sinh-Gordon (DSHG) parity invariant system

- $$V(x) = [\xi \cosh(2x) - M]^2$$

  where ξ and M are positive real parameters and $\lim_{|x|\to\infty} V(x) = \infty$.

The double sinh-Gordon (DSHG) parity invariant system

- $$V(x) = [\xi \cosh(2x) - M]^2$$
  where ξ and M are positive real parameters and $\lim_{|x|\to\infty} V(x) = \infty$.
- The change of independent variable $z = e^{2x}$ and
  $$\psi(z) = z^{\frac{1-M}{2}} \exp\left[-\frac{\xi}{4}\left(z + \frac{1}{z}\right)\right] \phi(z)$$

The double sinh-Gordon (DSHG) parity invariant system

- $$V(x) = [\xi \cosh(2x) - M]^2$$

  where ξ and M are positive real parameters and $\lim_{|x|\to\infty} V(x) = \infty$.
- The change of independent variable $z = e^{2x}$ and

  $$\psi(z) = z^{\frac{1-M}{2}} \exp\left[-\frac{\xi}{4}\left(z + \frac{1}{z}\right)\right] \phi(z)$$

- The baseline condition $F_1(n) = 2n\xi - 2\xi(M-1) = 0 \Longrightarrow n = M - 1$,

  $$F_1(k) = 2\xi(k-n), \quad F_0(k) = 4k(n-k) + c_0(n), \quad F_{-1}(k) = -2k\xi,$$

  where $c_0(n) = 2n + 1 - E + \xi^2$.

DSHG - II

- The baseline condition $F_1(n) = 2n\xi - 2\xi(M-1) = 0 \Longrightarrow n = M-1$,
$$F_1(k) = 2\xi(k-n), \quad F_0(k) = 4k(n-k) + c_0(n), \quad F_{-1}(k) = -2k\xi,$$
where $c_0(n) = 2n + 1 - E + \xi^2$.

DSHG - II

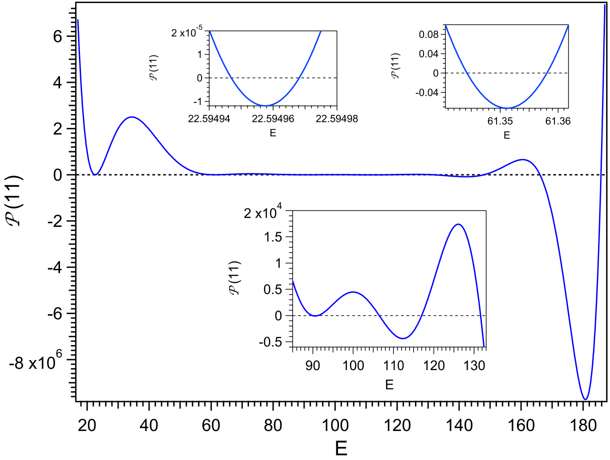
- The baseline condition $F_1(n) = 2n\xi - 2\xi(M-1) = 0 \Longrightarrow n = M-1$,

  $$F_1(k) = 2\xi(k-n), \quad F_0(k) = 4k(n-k) + c_0(n), \quad F_{-1}(k) = -2k\xi,$$

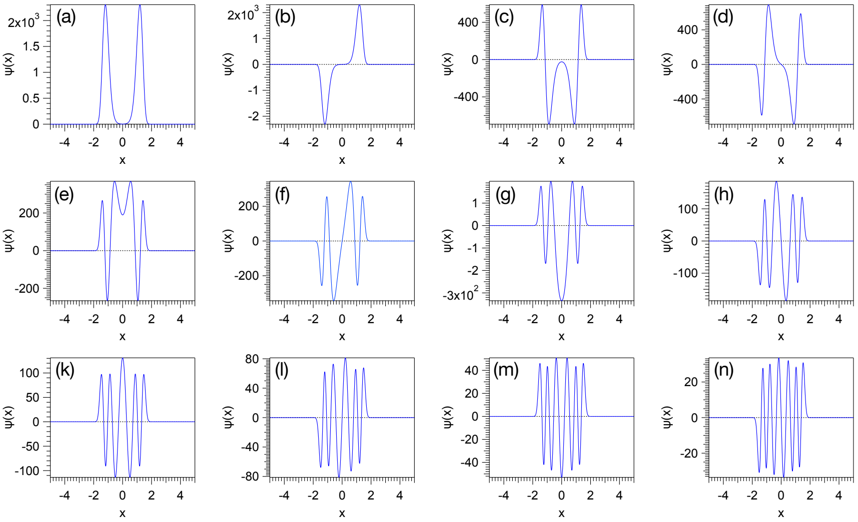
  where $c_0(n) = 2n + 1 - E + \xi^2$.

- $$\begin{aligned} F_1(n-1)P_{n1} &= -F_0(n)P_{n0} \\ \vdots \qquad & \qquad \vdots \qquad \vdots \qquad \vdots \\ F_1(n-k)P_{n,k} &= -F_{-1}(n+2-k)P_{n,k-2} - F_0(n+1-k)P_{n,k-1} \\ \vdots \qquad & \qquad \vdots \qquad \vdots \qquad \vdots \\ F_1(0)P_{nn} &= -F_{-1}(2)P_{n,n-2} - F_0(1)P_{n,n-1} \\ \mathcal{P}(n) &= F_{-1}(1)P_{n,n-1} + F_0(1)P_{nn} = b_0 P_{n,n-1} + c_0 P_{nn} \end{aligned}$$

  defines for nth baseline a finite orthogonal polynomial system $\{P_{nk},\, k = 0, 1, 2, \ldots, n,\, \mathcal{P}(n)\}$ in $E \Longrightarrow$ only simple roots



Constraint polynomial for the unperturbed DSHG with $\xi = 2$ on the 11th baseline corresponding to $M = 12$ is shown to have the maximum number of 12 *simple real* roots $E = 22.5949,\ 22.5949,\ 61.3442,\ 61.3580, 89.8744,\ 91.2808,$ $106.478,\ 117.007,\ 131.616,\ 147.980,\ 166.091,\ 185.777$ reproducing the results of Tab. 3 of Ref. Baradaran and Panahi, arXiv:1712.06439.



Interlaced even and odd parity polynomial eigenfunctions for the unperturbed DSHG with fixed $\xi = 2$ and $n = 10$ corresponding to the twelve *simple real* roots $E = 22.5949$, 22.5949, 61.3442, 61.3580, 89.8744, 91.2808, 106.478, 117.007, 131.616, 147.980, 166.091, 185.777 of the constraint polynomial.

Conclusions

- The concept of constraint polynomials provides a deeper reason for the existence of the original Kus polynomials and yields a versatile and efficient tool to determine polynomial eigenvalues and eigenfunctions.

Conclusions

- The concept of constraint polynomials provides a deeper reason for the existence of the original Kus polynomials and yields a versatile and efficient tool to determine polynomial eigenvalues and eigenfunctions.
- Our constraint polynomials, which differ from the weak orthogonal Bender-Dunne polynomials, are yet another class of polynomials closely related to the spectrum of quasi-exactly solvable models. For the models considered here, constraint polynomials terminated a finite chain of orthogonal polynomials characterized by a positive-definite moment functional $\mathcal{L}$, implying that a corresponding constraint polynomial has only real and simple zeros.

Conclusions

- The concept of constraint polynomials provides a deeper reason for the existence of the original Kus polynomials and yields a versatile and efficient tool to determine polynomial eigenvalues and eigenfunctions.
- Our constraint polynomials, which differ from the weak orthogonal Bender-Dunne polynomials, are yet another class of polynomials closely related to the spectrum of quasi-exactly solvable models. For the models considered here, constraint polynomials terminated a finite chain of orthogonal polynomials characterized by a positive-definite moment functional $\mathcal{L}$, implying that a corresponding constraint polynomial has only real and simple zeros.
- General constraint polynomial approach is shown to replace a set of algebraic equations of the functional Bethe Ansatz method by a single polynomial constraint.

Acknowledgments

Andrey Miroshnichenko, B. M. Rodríguez-Lara

Further reading

- AM, JPA 51, 295201 (2018)

Further reading

- AM, JPA 51, 295201 (2018)
- AM and Andrey E. Miroshnichenko, Constraint polynomial approach - an alternative to the functional Bethe Ansatz method? (arXiv:1712.06439)