```
In[1]:= F1[n_] := n (n - 1) a3 + n b2 + c1;
F0[n_] := n (n - 1) a2 + n b1 + c0;
Fm1[n_] := n (n - 1) a1 + n b0;
```

```
In[4]:= P[n_, -1] := 0;
P[n_, 0] := 1;
P[n_, 1] := - F0[n]/F1[n - 1] P[n, 0];
P[n_, k_] := - (Fm1[n + 2 - k] P[n, k - 2] + F0[n + 1 - k] P[n, k - 1])/F1[n - k];
Pp[n_] := Fm1[1] P[n, n - 1] + F0[0] P[n, n];
S[n_, z_] := Sum[P[n, n - k] z^k, {k, 0, n}];
```

The modified Manning potentials with three parameters

Even parity solutions

```
a3 = 0;
a2 = 4;
a1 = -4;
b2 = 4 √V1 ;
b1 = 6 + 4 √-λ - 4 √V1 ;
b0 = -2;
c1 = V1 + V2 + 3 √V1 + 2 √V1 √-λ ;
c0 = √-λ - √V1 - λ - V1 - V2 - V3;
```

```
En = Solve[F1[n] == 0, λ] // Simplify
```

$$\left\{\left\{\lambda \to -\frac{\left(\left(3+4\,n\right)\sqrt{V1}+V1+V2\right)^2}{4\,V1}\right\}\right\}$$

```
√-λ /. En /. {V1 → 1, V2 → -50, n → 10}
```

$\{3\}$

```
F1[k] /. En[[1]] // FullSimplify
F0[k] - c0 /. En[[1]] // FullSimplify
Fm1[k] /. En[[1]] // FullSimplify
```

$$V1+V2+\sqrt{V1}\left(3+4\,k+\sqrt{\frac{\left(\left(3+4\,n\right)\sqrt{V1}+V1+V2\right)^2}{V1}}\right)$$

$$2\,k\left(1+2\,k-2\sqrt{V1}+\sqrt{\frac{\left(\left(3+4\,n\right)\sqrt{V1}+V1+V2\right)^2}{V1}}\right)$$

$$2\left(1-2\,k\right)k$$

```
(4 n + 3) √V1 + V1 /. n → 10 /. V1 → 1
```

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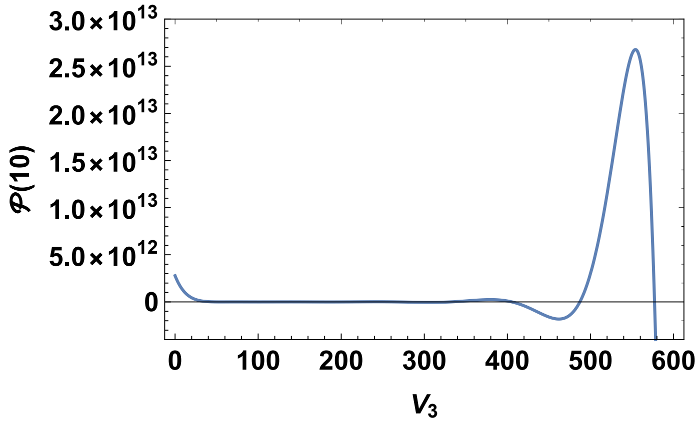
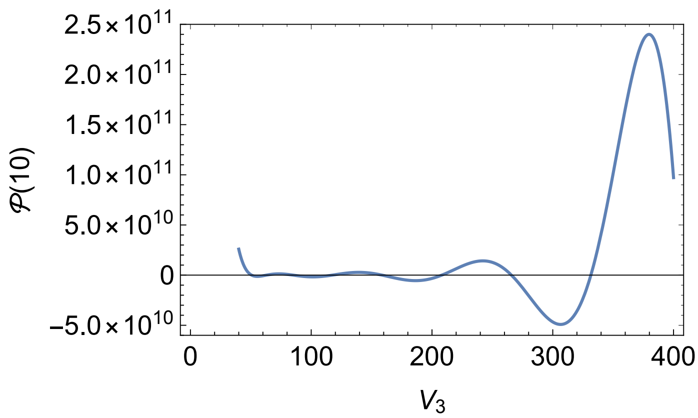
```
P10 = Pp[10] /. En[[1]] /. n → 10;
```

```
Fig1 = Plot[P10 /. {V1 → 1, V2 → -50}, {V3, 0, 600}, Frame → True,
  FrameLabel → {"V3", "𝒫(10)"}, FrameStyle → Directive[14],
  PlotRange → {-4 × 10^12, 3 × 10^13}, FrameTicksStyle → Larger,
  BaseStyle → {FontWeight → "Bold", FontSize → 18}]
```

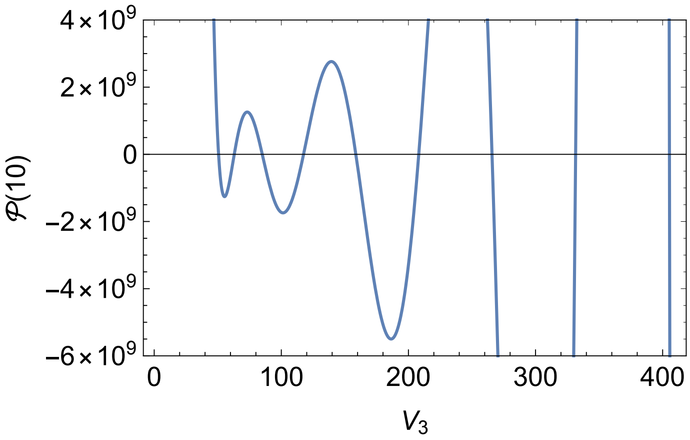



```
mydata = Flatten[Cases[Fig1, Line[x__] → x, Infinity], 1];
Export["/Users/andreym/Desktop/Hatami_Fig1.dat",
  Join[{"Xx Yy"}, mydata], "Table"];
```

```
Plot[P10 /. {V1 → 1, V2 → -50}, {V3, 40, 400},
 Frame → True, FrameLabel → {"V3", "𝒫(10)"},
 FrameStyle → Directive[14], PlotRange → {-6 × 10^10, 2.5 × 10^11}]
```



```
Plot[P10 /. {V1 → 1, V2 → -50}, {V3, 40, 410},
 Frame → True, FrameLabel → {"V₃", "𝒫(10)"},
 FrameStyle → Directive[14], PlotRange → {-6 × 10^9, 4 × 10^9}]
```



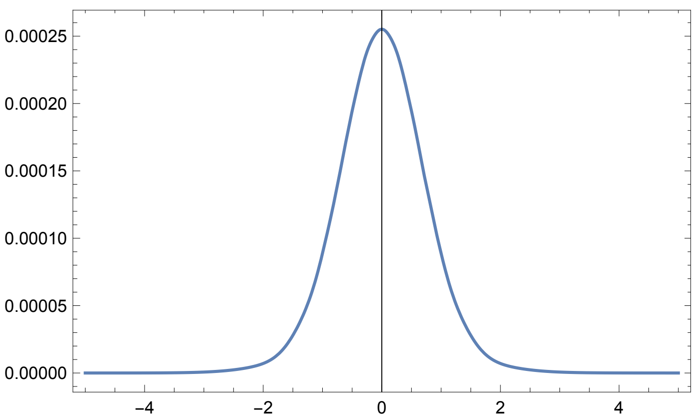

```
sol = Solve[{P10 /. {V1 → 1, V2 → -50}} == 0, V3] // N // Sort
{{V3 → 50.6499}, {V3 → 62.9912}, {V3 → 85.016},
 {V3 → 117.499}, {V3 → 158.65}, {V3 → 208.126}, {V3 → 265.78},
 {V3 → 331.54}, {V3 → 405.368}, {V3 → 487.239}, {V3 → 577.141}}
```

Wave function

```
√(-λ) /. En[[1]] /. {V1 → 1, V2 → -50} /. n → 10
3
```

$$\psi[x\_] = \mathrm{Exp}\left[\frac{\sqrt{V1}}{2} z\right] (1-z)^{\frac{\sqrt{-\lambda}}{2}} S[10, z] \text{ /. } z \to \mathrm{Tanh}[x]^2 \text{ /. En[[1]] /. }$$

```
    {V1 → 1, V2 → -50} /. n → 10;
Wave[i_] := Plot[ψ[x] /. sol[[i]], {x, -5, 5}, PlotRange → All, Frame → True]
Wave[11]
```



```
For[i = 1, i ≤ Length[sol], i = i + 1,
 wave = Flatten[Cases[Wave[i], Line[x__] → x, Infinity], 1];
 xs = StringJoin["x", ToString[i], " w", ToString[i]];
Export[StringJoin["/Users/andreym/Desktop/Hatami_Wave1_s",
   ToString[i], ".dat"], Join[{xs}, wave], "Table"];
]
```

Odd parity solutions

```
a3 = 0;
a2 = 4;
a1 = -4;
```

$b2 = 4\sqrt{V1};$

$b1 = 10 + 4\sqrt{-\lambda} - 4\sqrt{V1};$

$b0 = -6;$

$c1 = V1 + V2 + 5\sqrt{V1} + 2\sqrt{V1}\sqrt{-\lambda};$

$c0 = 3\sqrt{-\lambda} - 3\sqrt{V1} - \lambda - V1 - V2 - V3 + 2;$

```
En = Solve[F1[n] == 0, λ] // Simplify
```

$$\left\{\left\{\lambda \to -\frac{\left((5+4n)\sqrt{V1} + V1 + V2\right)^2}{4\,V1}\right\}\right\}$$

```
F1[k] /. En[[1]] // FullSimplify
F0[k] - c0 /. En[[1]] // FullSimplify
Fm1[k] /. En[[1]] // FullSimplify
```

$$V1 + V2 + \sqrt{V1}\left(5 + 4k + \sqrt{\frac{\left((5+4n)\sqrt{V1} + V1 + V2\right)^2}{V1}}\right)$$

$$2k\left(3 + 2k - 2\sqrt{V1} + \sqrt{\frac{\left((5+4n)\sqrt{V1} + V1 + V2\right)^2}{V1}}\right)$$

$$-2k(1 + 2k)$$

$(4n + 5)\sqrt{V1} + V1$ /. n → 10 /. V1 → 1

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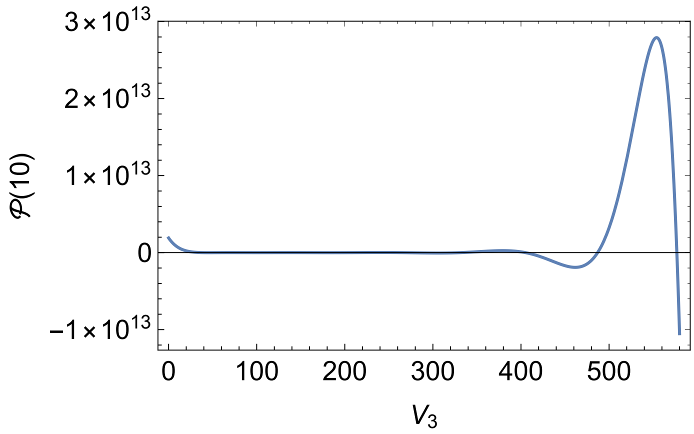
$\sqrt{-\lambda}$ /. En /. {V1 → 1, V2 → -50, n → 10}

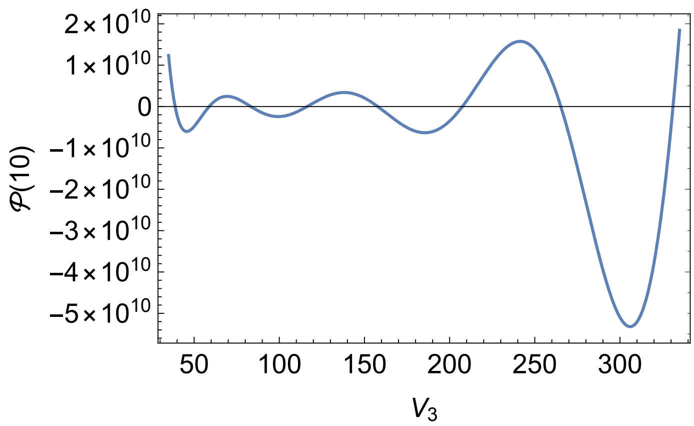
{2}

```
P10 = Pp[10] /. En[[1]] /. n → 10;
```

```
Fig2 = Plot[P10 /. {V1 → 1, V2 → -50}, {V3, 0, 580}, Frame → True,
  FrameLabel → {"V3", "𝒫(10)"}, FrameStyle → Directive[14], PlotRange → All]
```




```
Plot[P10 /. {V1 → 1, V2 → -50}, {V3, 35, 335}, Frame → True,
 FrameLabel → {"V3", "𝒫(10)"}, FrameStyle → Directive[14], PlotRange → All]
```




```
sol = Solve[{P10 /. {V1 → 1, V2 → -50}} == 0, V3] // N
{{V3 → 38.8277}, {V3 → 58.8256}, {V3 → 83.2712},
 {V3 → 116.335}, {V3 → 157.819}, {V3 → 207.504}, {V3 → 265.299},
 {V3 → 331.158}, {V3 → 405.056}, {V3 → 486.981}, {V3 → 576.924}}
```

```
mydata = Flatten[Cases[Fig2, Line[x__] → x, Infinity], 1];
Export["/Users/andreym/Desktop/Hatami_Fig2.dat",
  Join[{"Xx Yy"}, mydata], "Table"];
```

Wave function

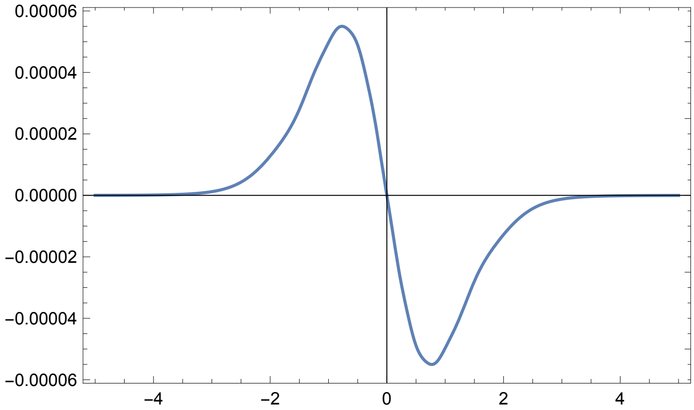
$\sqrt{-\lambda}$ /. En[[1]] /. {V1 → 1, V2 → -50} /. n → 10

2

ψ[x_] = Exp$\left[\frac{\sqrt{V1}}{2} z\right]$ $(1-z)^{\frac{\sqrt{-\lambda}}{2}}$ Tanh[x] S[10, z] /. z → Tanh[x]^2 /. En[[1]] /.
  {V1 → 1, V2 → -50} /. n → 10;

```
Wave[i_] := Plot[ψ[x] /. sol[[i]], {x, -5, 5}, PlotRange → All, Frame → True]
```

```
Wave[11]
```




```
For[i = 1, i ≤ Length[sol], i = i + 1,
 wave = Flatten[Cases[Wave[i], Line[x__] → x, Infinity], 1];
 xs = StringJoin["x", ToString[i], " w", ToString[i]];
Export[StringJoin["/Users/andreym/Desktop/Hatami_Wave2_s",
   ToString[i], ".dat"], Join[{xs}, wave], "Table"];
]
```

modified Manning potentials with three parameters

Even parity solutions

$$M2 = \left\{\lambda 1 \to \frac{1}{4}\left(1+\sqrt{1-4\,V1}\right),\ \lambda 2 \to \frac{1}{2}\left(1-\sqrt{1+\frac{V3}{1+g}}\right)\right\};$$

$$a3 = 1;$$

$$a2 = -2 - \frac{1}{g};$$

$$a1 = 1 + \frac{1}{g};$$

$$b2 = 2\,\lambda 1 + 2\,\lambda 2 + 1;$$

$$b1 = -1 - \frac{1}{2\,g} - \left(2\,\lambda 1 + \frac{1}{2}\right)\left(1 + \frac{1}{g}\right) - 2\,\lambda 2;$$

$$b0 = \frac{1+g}{g};$$

$$c1 = (\lambda 1 + \lambda 2)^2 + \frac{Em}{4};$$

$$c0 = -\frac{1+g}{g}\left(2\,\lambda 1 + \frac{2\,\lambda 2\,g - V2}{1+g} - V1 - \frac{V3}{(1+g)^2} + Em\right);$$

```
En = Solve[F1[n] == 0, Em] // Simplify
```

$$\{\{Em \to -4\,(n + \lambda 1 + \lambda 2)^2\}\}$$

```
F1[k] /. En[[1]] // FullSimplify
F0[k] - c0 /. En[[1]] // FullSimplify
Fm1[k] /. En[[1]] // FullSimplify
```

$(k - n)\ (k + n + 2\ (\lambda 1 + \lambda 2))$

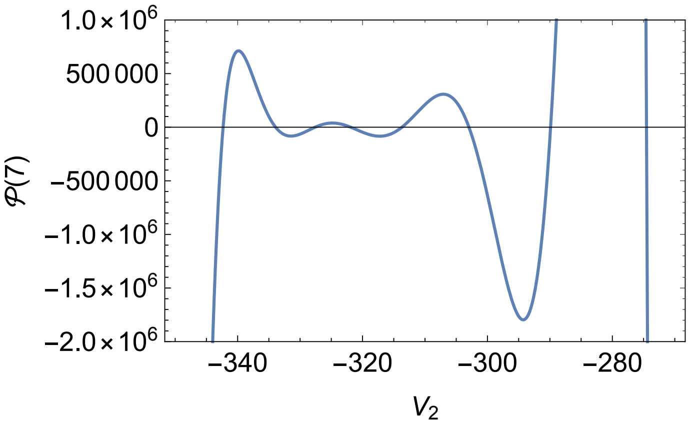
$-\frac{k\ (2\ (k + 2\ \lambda 1) + g\ (-1 + 4\ k + 4\ \lambda 1 + 4\ \lambda 2))}{2\ g}$

$\frac{(1 + g)\ k^2}{g}$

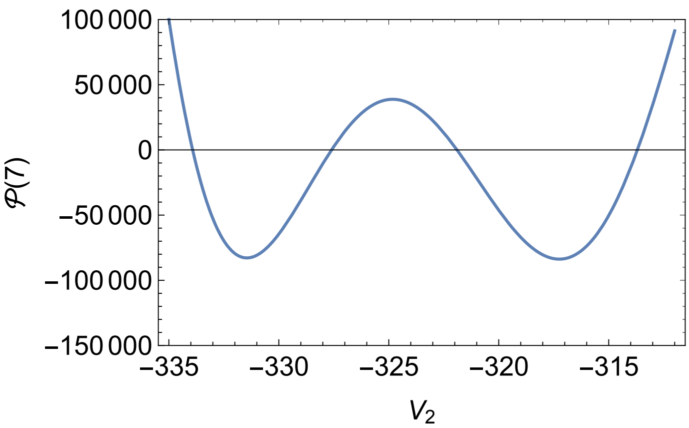
```
c0 /. En[[1]] // Simplify
```

$-\frac{(1 + g)\ \left(-V1 - \frac{V3}{(1+g)^2} + 2\ \lambda 1 - 4\ (n + \lambda 1 + \lambda 2)^2 - \frac{V2-2\ g\ \lambda 2}{1+g}\right)}{g}$

```
P7 = Pp[7] /. En[[1]] /. n → 7 /. M2;
Fig3 = Plot[P7 /. {V1 → 0.09, V3 → 400, g → 1/4}, {V2, -350, -270},
  PerformanceGoal → "Quality", Frame → True, FrameLabel → {"V2", "𝒫(7)"},
  FrameStyle → Directive[14], PlotRange → {-2 000 000, 1 000 000}]
```




```
Plot[P7 /. {V1 → 0.09, V3 → 400, g → 1/4}, {V2, -335, -312},
 PerformanceGoal → "Quality", Frame → True, FrameLabel → {"V2", "𝒫(7)"},
 FrameStyle → Directive[14], PlotRange → {{-150 000, 100 000}}]
```



```
sol = Solve[{P7 /. {V1 → 0.09, V3 → 400, g → 1/4}} == 0, V2] // Sort
```

```
{{V2 → -342.323}, {V2 → -333.92}, {V2 → -327.6}, {V2 → -321.903},
 {V2 → -313.691}, {V2 → -302.958}, {V2 → -289.886}, {V2 → -274.53}}
```

```
mydata = Flatten[Cases[Fig3, Line[x__] → x, Infinity], 1];
Export["/Users/andreym/Desktop/Hatami_Fig3.dat",
  Join[{"Xx Yy"}, mydata], "Table"];
```

Wave function

```
λ1 + λ2 + n /. M2 /. En[[1]] /. {V1 → 0.09, V3 → 400, g → 1/4} /. n → 7
```

```
-1.00824
```

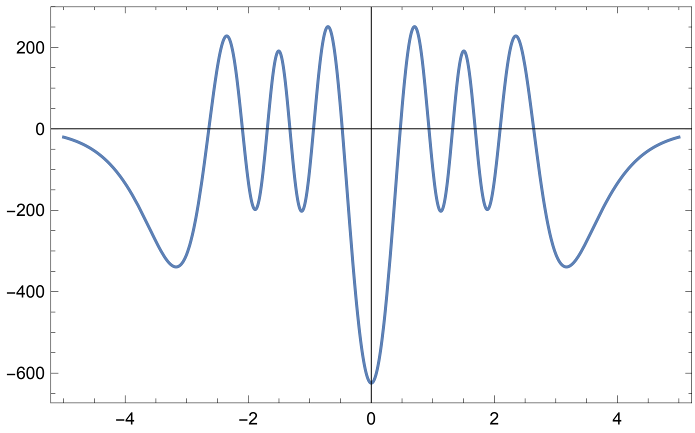
```
Solve[(λ1 + λ2 + n /. M2 /. En[[1]] /. {V1 → 0.09, g → 1/4} /. n → 7) == 0, V3]
```

```
{{V3 → 314.763}}
```

```
ψ[x_] = Cosh[x]^(2 λ1) (1 + g Cosh[x]^2)^λ2 S[7, z] /. z → -Sinh[x]^2 /. En[[1]] /. M2 /.
    {V1 → 0.09, V3 → 400, g → 1/4} /. n → 7;
```

```
Wave[i_] := Plot[ψ[x] /. sol[[i]], {x, -5, 5}, PlotRange → All, Frame → True]
```

```
Wave[2]
```




```
For[i = 1, i ≤ Length[sol], i = i + 1,
 wave = Flatten[Cases[Wave[i], Line[x__] → x, Infinity], 1];
 xs = StringJoin["x", ToString[i], " w", ToString[i]];
Export[StringJoin["/Users/andreym/Desktop/Hatami_Wave3_s",
   ToString[i], ".dat"], Join[{xs}, wave], "Table"];
]
```

Electron in Coulomb and magnetic fields and relative motion of two electrons in an external oscillator potential

```
a3 = 0;
a2 = 0;
a1 = 1;
b2 = -1;
b1 = 0;
b0 = 2 γ;
c1 = ϵ;
c0 = b;
```

```
En = Solve[F1[n] == 0, ϵ] // Simplify
```

{{ϵ → n}}

```
F1[k] /. En[[1]] // FullSimplify
F0[k] /. En[[1]] // FullSimplify
Fm1[k] /. En[[1]] // FullSimplify
```

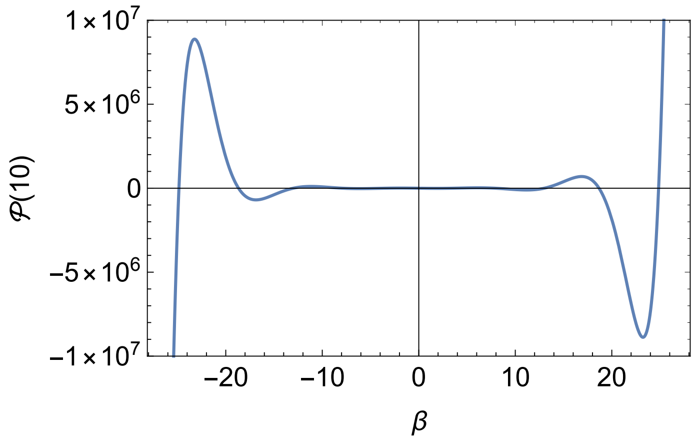
$-k + n$

b

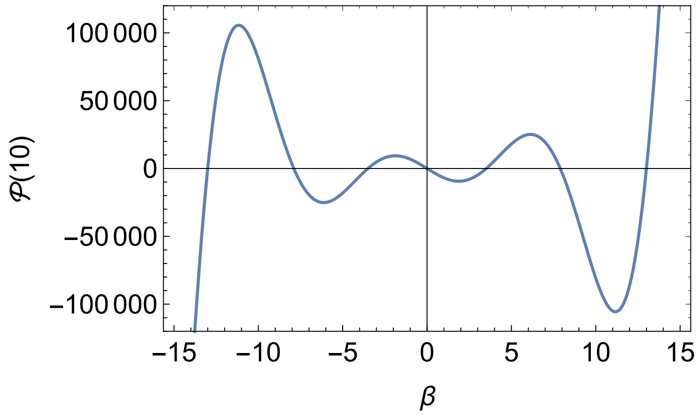
$k\,(-1 + k + 2\,\gamma)$

```
P10 = Pp[10] /. En[[1]] /. n → 10;
```

```
Fig4 = Plot[P10 /. {γ → 0.5}, {b, -27, 27},
  PlotPoints → 40, Frame → True, FrameLabel → {"β", "𝒫(10)"},
  FrameStyle → Directive[14], PlotRange → {-1 × 10^7, 1 × 10^7}]
```



```
Plot[P10 /. {γ → 0.5}, {b, -15, 15},
 PlotPoints → 40, Frame → True, FrameLabel → {"β", "𝒫(10)"},
 FrameStyle → Directive[14], PlotRange → {-1.2 × 10^5, 1.2 × 10^5}]
```




```
sol = Solve[{P10 /. {γ → 0.5}} == 0, b, WorkingPrecision → 20] // Chop
```

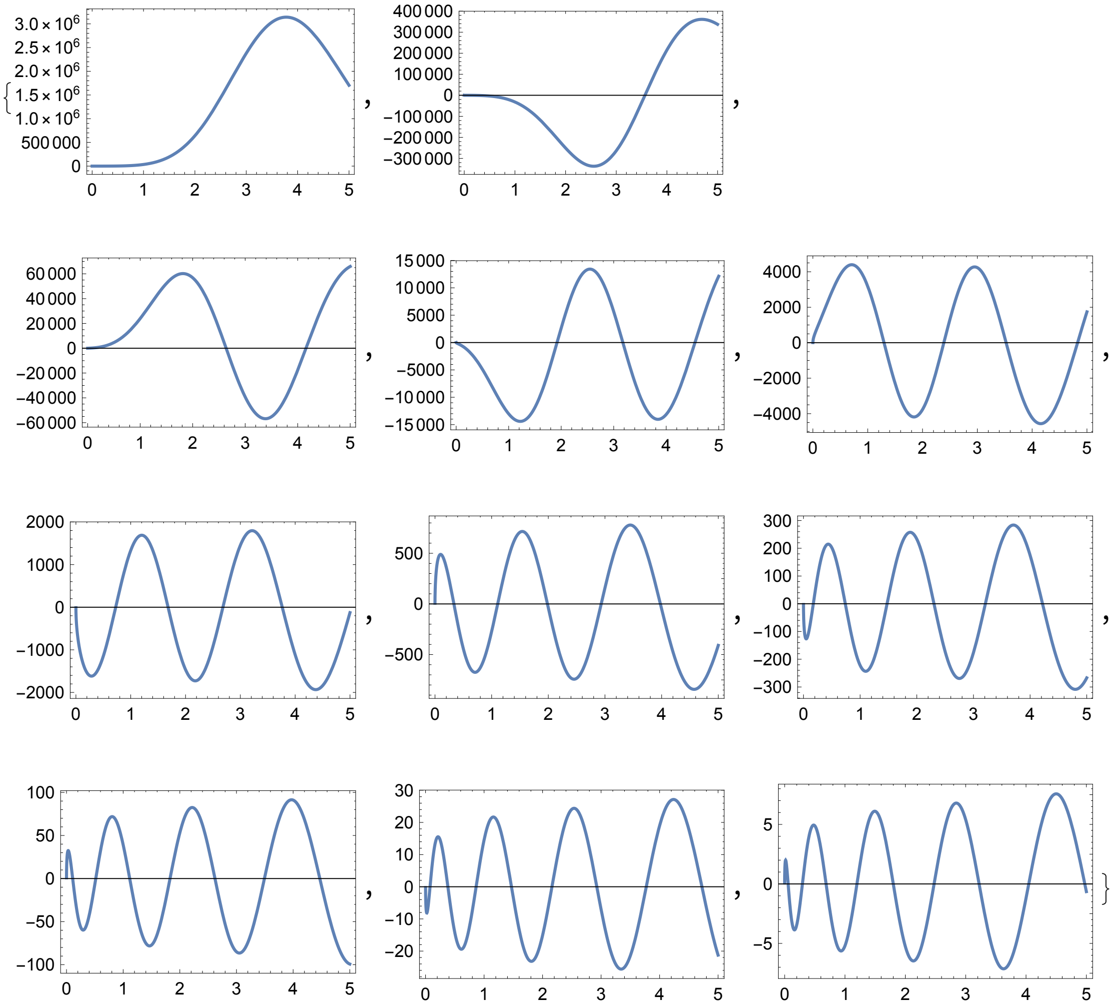
```
{{b → -24.8502}, {b → -18.676}, {b → -13.0012}, {b → -7.89603}, {b → -3.50671},
 {b → 0}, {b → 3.50671}, {b → 7.89603}, {b → 13.0012}, {b → 18.676}, {b → 24.8502}}
```

```
mydata = Flatten[Cases[Fig4, Line[x__] → x, Infinity], 1];
Export["/Users/andreym/Desktop/Hatami_Fig4.dat",
  Join[{"Xx Yy"}, mydata], "Table"];
```

Wave function

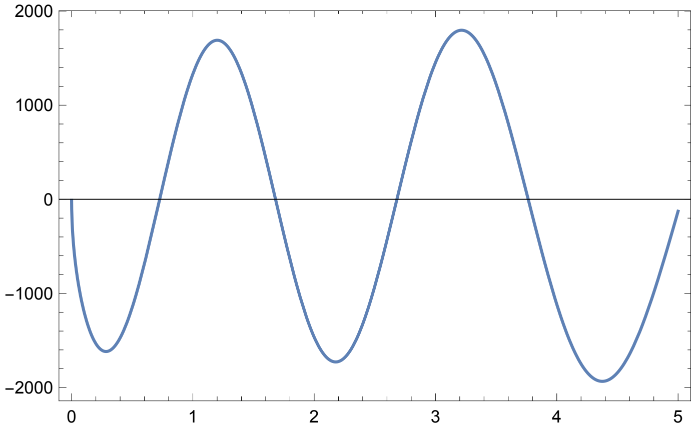
```
ψ[x_] = x^γ Exp[-x^2/4] S[10, x] /. En[[1]] /. {γ → 0.5} /. n → 10;
```

```
Table[Plot[ψ[x] /. sol[[i]], {x, -5, 5}, PlotRange → All, Frame → True], {i, 1, 11}]
```




```
Wave[i_] := Plot[ψ[x] /. sol[[i]], {x, -5, 5}, PlotRange → All, Frame → True]
```

```
Wave[6]
```



```
For[i = 1, i ≤ Length[sol], i = i + 1,
 wave = Flatten[Cases[Wave[i], Line[x__] → x, Infinity], 1];
 xs = StringJoin["x", ToString[i], " w", ToString[i]];
Export[StringJoin["/Users/andreym/Desktop/Hatami_Wave_Coulomb",
   ToString[i], ".dat"], Join[{xs}, wave], "Table"];
]
```

The hyperbolic Razavy potential

```
a3 = 0;
a2 = 4;
a1 = -4;
b2 = -4 ξ;
b1 = 4 (α + β + ξ + 1);
b0 = -2 (2 α + 1);
c1 = 2 ξ (Nm - α - β);
c0 = Em + (α + β)^2 + ξ (2 α - Nm);
```

```
En = Solve[F1[n] == 0, Nm] // Simplify
```

$\{\{Nm \to 2 n + \alpha + \beta\}\}$

```
F1[k] /. En[[1]] // FullSimplify
F0[k] - c0 /. En[[1]] // FullSimplify
Fm1[k] /. En[[1]] // FullSimplify
```

$4 (-k + n) \xi$

$4 k (k + \alpha + \beta + \xi)$

$-2 k (-1 + 2 k + 2 \alpha)$

```
c0 /. En[[1]] // FullSimplify
```

$Em + (\alpha + \beta)^2 + (-2 n + \alpha - \beta) \xi$

```
P0 = Pp[0] /. En[[1]] /. n → 0 /. {α → 0, β → 0}
```

Em

```
Solve[P0 == 0, Em]
```

$\{\{Em \to 0\}\}$

```
P1 = Pp[1] /. En[[1]] /. n → 1 /. {α → 0, β → 0}
```

$$-2 - \frac{(Em - 2 \xi) (Em - 2 \xi + 4 (1 + \xi))}{4 \xi}$$

```
Solve[P1 == 0, Em]
```

$\left\{\left\{Em \to 2 \left(-1 - \sqrt{1 + \xi^2}\right)\right\}, \left\{Em \to 2 \left(-1 + \sqrt{1 + \xi^2}\right)\right\}\right\}$

```
4 ξ z1 - 4 (1 + ξ/2) /. z1 → (ξ + 1 - Sqrt[ξ^2 + 1])/(2 ξ) // Simplify
```

$-2\left(1+\sqrt{1+\xi^2}\right)$

```
P0 = Pp[0] /. En[[1]] /. n → 0 /. {α → 1, β → 0}
```

$1+\text{Em}+\xi$

```
Solve[P0 == 0, Em]
```

$\{\{\text{Em}\to -1-\xi\}\}$

```
P1 = Pp[1] /. En[[1]] /. n → 1 /. {α → 1, β → 0}
```

$-6-\frac{(1+\text{Em}-\xi)\,(1+\text{Em}-\xi+4\,(2+\xi))}{4\,\xi}$

```
Solve[P1 == 0, Em]
```

$\left\{\left\{\text{Em}\to -5-\xi-2\sqrt{4-2\,\xi+\xi^2}\right\},\left\{\text{Em}\to -5-\xi+2\sqrt{4-2\,\xi+\xi^2}\right\}\right\}$

```
4 ξ z1 - 1 + ξ - 4 (2 + ξ/2) /. z1 → (ξ + 2 - Sqrt[ξ^2 + 2 ξ + 4])/(2 ξ) // Simplify
```

$-5+\xi-2\sqrt{4+2\,\xi+\xi^2}$

```
P0 = Pp[0] /. En[[1]] /. n → 0 /. {α → 0, β → 1}
```

$1+\text{Em}-\xi$

```
Solve[P0 == 0, Em]
```

$\{\{\text{Em}\to -1+\xi\}\}$

```
P1 = Pp[1] /. En[[1]] /. n → 1 /. {α → 0, β → 1}
```

$-2-\frac{(1+\text{Em}-3\,\xi)\,(1+\text{Em}-3\,\xi+4\,(2+\xi))}{4\,\xi}$

```
Solve[P1 == 0, Em]
```

$\left\{\left\{\text{Em}\to -5+\xi-2\sqrt{4+2\,\xi+\xi^2}\right\},\left\{\text{Em}\to -5+\xi+2\sqrt{4+2\,\xi+\xi^2}\right\}\right\}$

```
P0 = Pp[0] /. En[[1]] /. n → 0 /. {α → 1, β → 1}
```

$4+\text{Em}$

```
Solve[P0 == 0, Em]
```

$\{\{\text{Em}\to -4\}\}$

```
P1 = Pp[1] /. En[[1]] /. n → 1 /. {α → 1, β → 1}
```

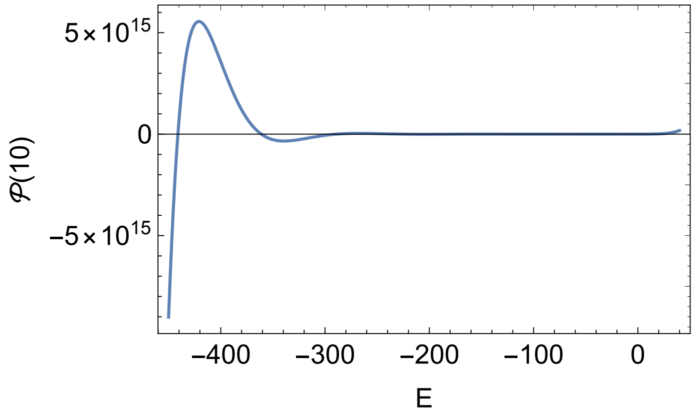
$-6-\frac{(4+\text{Em}-2\,\xi)\,(4+\text{Em}-2\,\xi+4\,(3+\xi))}{4\,\xi}$

```
Solve[P1 == 0, Em]
```

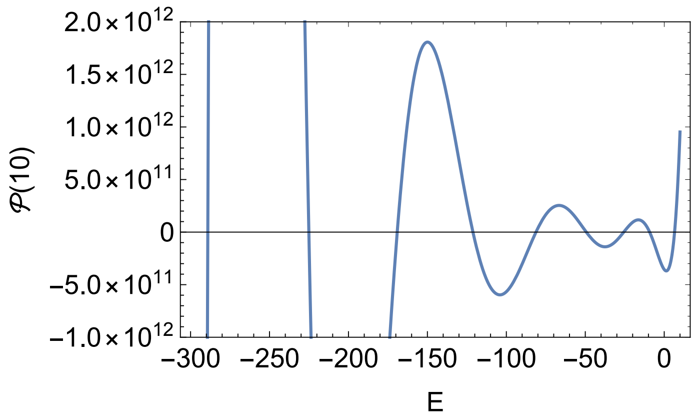
$\{\{Em \to 2\left(-5-\sqrt{9+\xi^2}\right)\}, \{Em \to 2\left(-5+\sqrt{9+\xi^2}\right)\}\}$

```
P10 = Pp[10] /. En[[1]] /. n → 10 /. {α → 0, β → 1};
```

```
Fig5 = Plot[P10 /. {ξ → 0.5}, {Em, -450, 40}, PlotPoints → 40, Frame → True,
  FrameLabel → {"E", "𝒫(10)"}, FrameStyle → Directive[14], PlotRange → All]
```




```
Plot[P10 /. {ξ → 0.5}, {Em, -300, 10},
 PlotPoints → 40, Frame → True, FrameLabel → {"E", "𝒫(10)"},
 FrameStyle → Directive[14], PlotRange → {-1 × 10^12, 2 × 10^12}]
```




```
sol = Solve[{P10 /. {ξ → 0.5}} == 0, Em, WorkingPrecision → 100] // N
```

```
{{Em → -441.066}, {Em → -361.073}, {Em → -289.084},
 {Em → -225.099}, {Em → -169.121}, {Em → -121.157}, {Em → -81.2206},
 {Em → -49.3476}, {Em → -25.6452}, {Em → -9.23983}, {Em → 6.55323}}
```

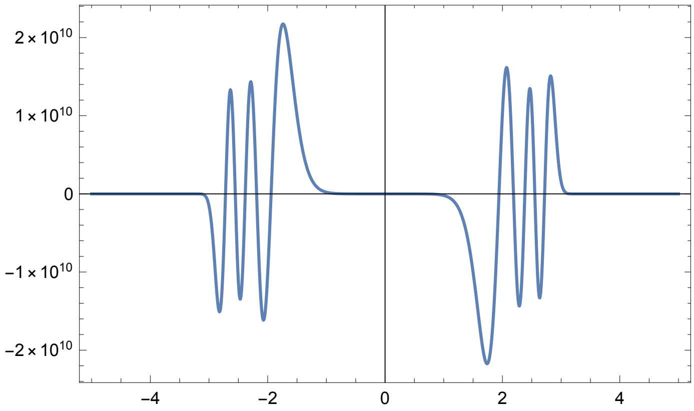
```
mydata = Flatten[Cases[Fig5, Line[x__] → x, Infinity], 1];
Export["/Users/andreym/Desktop/Hatami_Fig5.dat",
  Join[{"Xx Yy"}, mydata], "Table"];
```

Wave function

ψ[x_] = Exp$\left[-\frac{\xi}{4}\right.$ Cosh[2 x]$\left.\right]$ Cosh[x]$^\alpha$ Sinh[x]$^\beta$ S[10, z] /. z → Cosh[x]^2 /. En[[1]] /.
  {α → 0, β → 1} /. {ξ → 0.5} /. n → 10;

```
Wave[i_] := Plot[ψ[x] /. sol[[i]], {x, -5, 5}, PlotRange → All, Frame → True]
```

```
Wave[6]
```




```
For[i = 1, i ≤ Length[sol], i = i + 1,
 wave = Flatten[Cases[Wave[i], Line[x__] → x, Infinity], 1];
 xs = StringJoin["x", ToString[i], " w", ToString[i]];
Export[StringJoin["/Users/andreym/Desktop/Hatami_Wave4_s",
   ToString[i], ".dat"], Join[{xs}, wave], "Table"];
]
```

A double sinh-Gordon system

```
a3 = 0;
a2 = -4;
a1 = 0;
b2 = 2 ξ;
b1 = 4 M - 8;
b0 = -2 ξ;
c1 = -2 ξ (M - 1);
c0 = -Er - 1 + 2 M + ξ^2;
```

```
En = Solve[F1[n] == 0, M] // Simplify
```

{{M → 1 + n}}

```
F1[k] /. En[[1]] // FullSimplify
F0[k] - c0 /. En[[1]] // FullSimplify
Fm1[k] /. En[[1]] // FullSimplify
```

$2 (k - n) \xi$

$4 k (-k + n)$

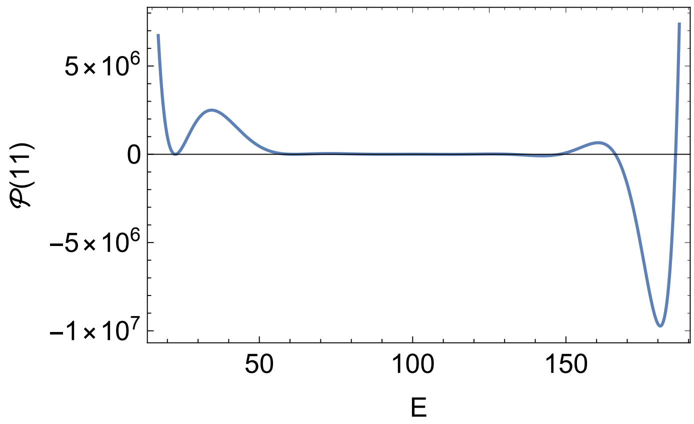
$-2 k \xi$

```
c0 /. En[[1]] // FullSimplify
```

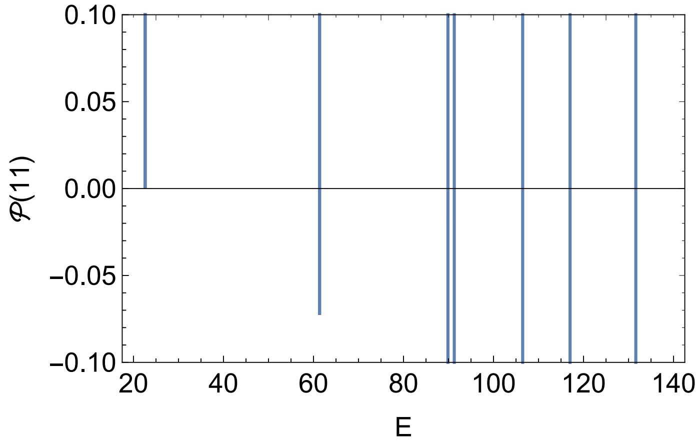
$1 - Er + 2 n + \xi^2$

```
P11 = Pp[11] /. En[[1]] /. n → 11;
```

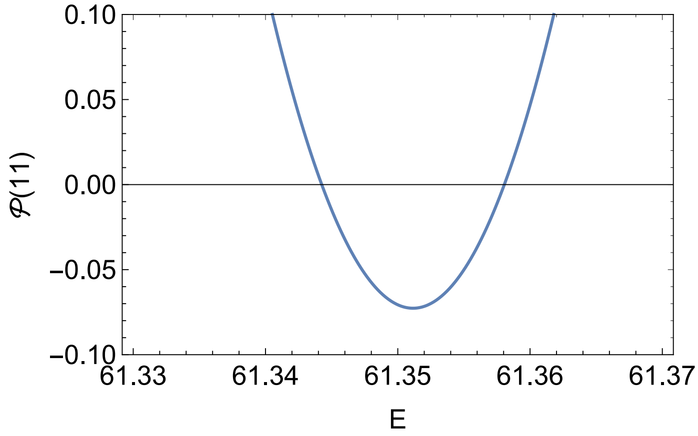
```
Fig6 = Plot[P11 /. {ξ → 2}, {Er, 17, 187}, Frame → True,
  FrameLabel → {"E", "𝒫(11)"}, FrameStyle → Directive[14], PlotRange → All]
```



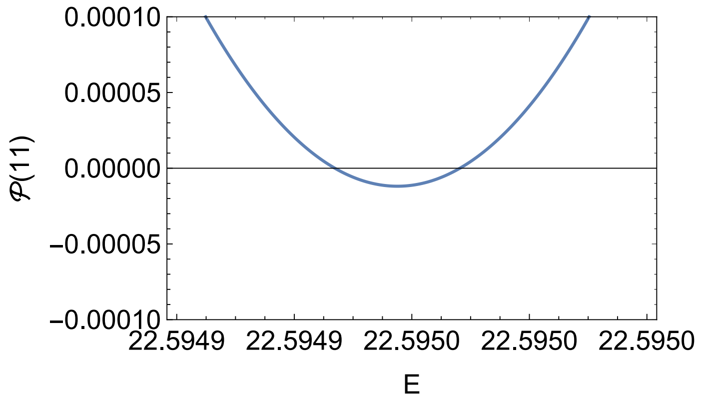

```
Plot[P11 /. {ξ → 2}, {Er, 20, 140}, Frame → True, FrameLabel → {"E", "𝒫(11)"},
 FrameStyle → Directive[14], PlotRange → {-0.1, 0.1}, PlotPoints → 10 000]
```




```
Fig6b = Plot[P11 /. {ξ → 2}, {Er, 61.33, 61.37}, Frame → True,
  FrameLabel → {"E", "𝒫(11)"}, FrameStyle → Directive[14],
  PlotRange → {-0.1, 0.1}, PlotPoints → 200]
```



```
Fig6c = Plot[P11 /. {ξ → 2}, {Er, 22.59492, 22.5950},
  Frame → True, FrameLabel → {"E", "𝒫(11)"}, FrameStyle → Directive[14],
  PlotRange → {-0.0001, 0.0001}, PlotPoints → 200]
```




```
sol = Solve[{P11 /. {ξ → 2}} == 0 , Er] // N // Sort
```

```
{{Er → 22.5949}, {Er → 22.595}, {Er → 61.3443}, {Er → 61.3581},
 {Er → 89.8745}, {Er → 91.2808}, {Er → 106.478}, {Er → 117.008},
 {Er → 131.617}, {Er → 147.981}, {Er → 166.092}, {Er → 185.778}}
```

```
mydata = Flatten[Cases[Fig6, Line[x__] → x, Infinity], 1];
Export["/Users/andreym/Desktop/Hatami_Fig6.dat",
  Join[{"Xx Yy"}, mydata], "Table"];
```

```
mydata = Flatten[Cases[Fig6b, Line[x__] → x, Infinity], 1];
Export["/Users/andreym/Desktop/Hatami_Fig6b.dat",
  Join[{"Xx Yy"}, mydata], "Table"];
```

```
mydata = Flatten[Cases[Fig6c, Line[x__] → x, Infinity], 1];
Export["/Users/andreym/Desktop/Hatami_Fig6c.dat",
  Join[{"Xx Yy"}, mydata], "Table"];
```

```
P5 = Pp[5] /. En[[1]] /. n → 5;
```

```
Solve[{P5 /. {ξ → 2}} == 0 , Er] // N // Sort // Chop
```

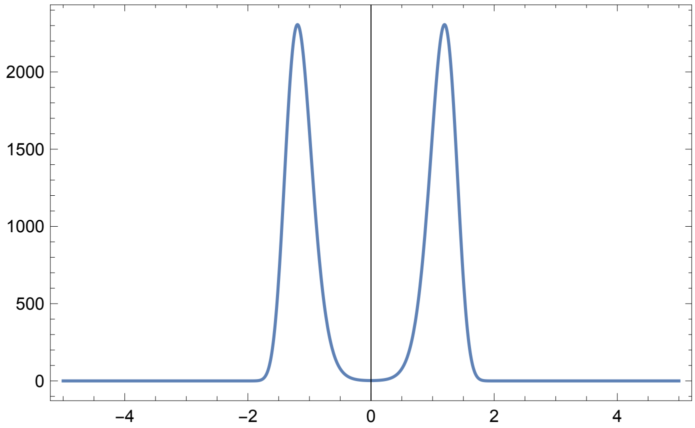
```
{{Er → 1.41047}, {Er → 2.4115}, {Er → 13.2917},
 {Er → 22.1448}, {Er → 34.2978}, {Er → 48.4437}}
```

Wave function

$$\psi[x\_] = z^{\frac{1-M}{2}}\,\mathrm{Exp}\left[-\frac{\xi}{4}\left(z+\frac{1}{z}\right)\right]\,S[11, z] \;/.\; z \to \mathrm{Exp}[2\,x] \;/.\; \mathrm{En}[[1]] \;/.\; \{\xi \to 2\} \;/.\; n \to 11;$$

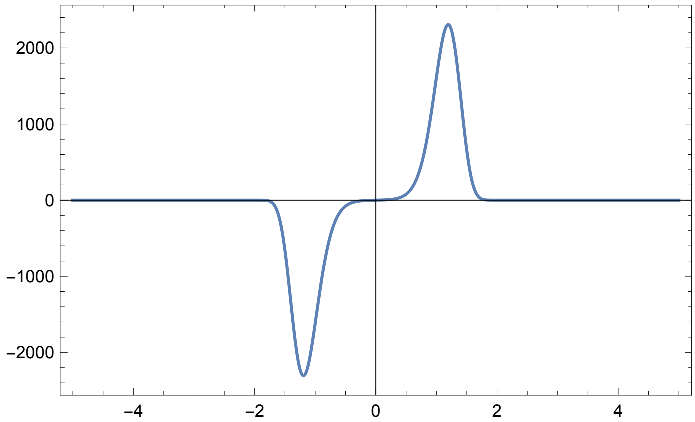
```
Wave[i_] := Plot[ψ[x] /. sol[[i]], {x, -5, 5}, PlotRange → All, Frame → True]
```

```
Wave[1]
sol[[1]]
```




```
{Er → 22.5949}
```

```
Wave[2]
sol[[2]]
```




```
{Er → 22.595}
```

```
For[i = 1, i ≤ Length[sol], i = i + 1,
 wave = Flatten[Cases[Wave[i], Line[x__] → x, Infinity], 1];
 xs = StringJoin["x", ToString[i], " w", ToString[i]];
Export[StringJoin["/Users/andreym/Desktop/Hatami_Wave5_s",
   ToString[i], ".dat"], Join[{xs}, wave], "Table"];
]
```

A perturbed double sinh-Gordon system

In[10]:=
```
a3 = 0;
a2 = 4;
a1 = -4;
b2 = -8 ξ;
b1 = 4 (α + β + 2 ξ + 1);
b0 = -2 (2 α + 1);
c1 = 4 ξ (M - α - β - 1);
c0 = Er - M^2 - ξ^2 + (α + β)^2 + 2 ξ (2 α - M + 1);
```

```
b1 // FullSimplify
```

$4\,(1+\alpha+\beta+2\,\xi)$

```
F1[n]
```

$-8\,n\,\xi+4\,(-1+M-\alpha-\beta)\,\xi$

In[18]:=
```
En = Solve[F1[n] == 0, M] // Simplify
```

Out[18]= $\{\{M\to 1+2\,n+\alpha+\beta\}\}$

```
F1[k] /. En[[1]] // FullSimplify
F0[k] - c0 /. En[[1]] // FullSimplify
Fm1[k] /. En[[1]] // FullSimplify
```

$8\,(-k+n)\,\xi$

$4\,k\,(k+\alpha+\beta+2\,\xi)$

$-2\,k\,(-1+2\,k+2\,\beta)$

```
c0 /. En[[1]] // FullSimplify
```

$Er+(\alpha+\beta)^2-(1+2\,n+\alpha+\beta)^2-2\,(2\,n-\alpha+\beta)\,\xi-\xi^2$

```
P0 = Pp[0] /. En[[1]] /. n → 0 /. {α → l + 1, β → 0} // Simplify
```

$-3+Er-2\,\xi-\xi^2-2\,l\,(1+\xi)$

```
Solve[P0 == 0, Er] // FullSimplify
```

$\{\{Er\to 3+2\,l\,(1+\xi)+\xi\,(2+\xi)\}\}$

```
P1 = Pp[1] /. En[[1]] /. n → 1 /. {α → l + 1, β → 0} // FullSimplify
```

$-2-\frac{1}{8\,\xi}\left(7-Er+(-2+\xi)\,\xi+2\,l\,(1+\xi)\right)\left(15-Er+2\,l\,(3+\xi)+\xi\,(6+\xi)\right)$

```
Solve[P1 == 0, Er] // FullSimplify
```

$\left\{\left\{Er\to 11+2\,l\,(2+\xi)+\xi\,(2+\xi)-2\sqrt{(2+l)^2+4\,(1+l)\,\xi+4\,\xi^2}\right\},\right.$
$\left.\left\{Er\to 11+2\,l\,(2+\xi)+\xi\,(2+\xi)+2\sqrt{(2+l)^2+4\,(1+l)\,\xi+4\,\xi^2}\right\}\right\}$

```
P0 = Pp[0] /. En[[1]] /. n → 0 /. {α → l + 1, β → 1} // Simplify
```

$-5+Er-\xi^2-2\,l\,(1+\xi)$

```
Solve[P0 == 0, Er] // FullSimplify
```

$\{\{Er \to 5 + \xi^2 + 2\, l\, (1 + \xi)\}\}$

```
P1 = Pp[1] /. En[[1]] /. n → 1 /. {α → l + 1, β → 1} // FullSimplify
```

$-\frac{1}{8\,\xi}\left(189 + Er^2 + 14\,\xi^2 + \xi^4 + 4\,l^2\,(1+\xi)\,(3+\xi) - 2\,Er\,(15 + \xi^2 + 2\,l\,(2+\xi)) + 4\,l\,(24 + \xi\,(11 + \xi\,(2+\xi)))\right)$

```
Solve[P1 == 0, Er] // FullSimplify
```

$\{\{Er \to 15 + \xi^2 + 2\,l\,(2+\xi) - 2\sqrt{(3+l)^2 + 4\,l\,\xi + 4\,\xi^2}\},$
$\{Er \to 15 + \xi^2 + 2\,l\,(2+\xi) + 2\sqrt{(3+l)^2 + 4\,l\,\xi + 4\,\xi^2}\}\}$

```
P0 = Pp[0] /. En[[1]] /. n → 0 /. {α → -l, β → 0} // Simplify
```

$-1 + Er - \xi^2 + 2\,l\,(1+\xi)$

```
Solve[P0 == 0, Er] // FullSimplify
```

$\{\{Er \to 1 + \xi^2 - 2\,l\,(1+\xi)\}\}$

```
P1 = Pp[1] /. En[[1]] /. n → 1 /. {α → -l, β → 0} // FullSimplify
```

$-\frac{1}{8\,\xi}\left(45 + Er^2 - 2\,\xi^2 + \xi^4 + 4\,l^2\,(1+\xi)\,(3+\xi) - 2\,Er\,(7 + \xi^2 - 2\,l\,(2+\xi)) - 4\,l\,(12 + \xi\,(3 + \xi\,(2+\xi)))\right)$

```
Solve[P1 == 0, Er] // FullSimplify
```

$\{\{Er \to 7 + \xi^2 - 2\,l\,(2+\xi) - 2\sqrt{(-1+l)^2 - 4\,l\,\xi + 4\,\xi^2}\},$
$\{Er \to 7 + \xi^2 - 2\,l\,(2+\xi) + 2\sqrt{(-1+l)^2 - 4\,l\,\xi + 4\,\xi^2}\}\}$

```
P0 = Pp[0] /. En[[1]] /. n → 0 /. {α → -l, β → 1} // Simplify
```

$-3 + Er + 2\,\xi - \xi^2 + 2\,l\,(1+\xi)$

```
Solve[P0 == 0, Er] // FullSimplify
```

$\{\{Er \to 3 + (-2+\xi)\,\xi - 2\,l\,(1+\xi)\}\}$

```
P1 = Pp[1] /. En[[1]] /. n → 1 /. {α → -l, β → 1} // FullSimplify
```

$-\frac{1}{8\,\xi}\left(105 + Er^2 + 4\,l^2\,(1+\xi)\,(3+\xi) - 4\,l\,(18 + 3\,\xi + \xi^3) + Er\,(-22 - 2\,(-2+\xi)\,\xi + 4\,l\,(2+\xi)) + \xi\,(-28 + \xi\,(10 + (-4+\xi)\,\xi))\right)$

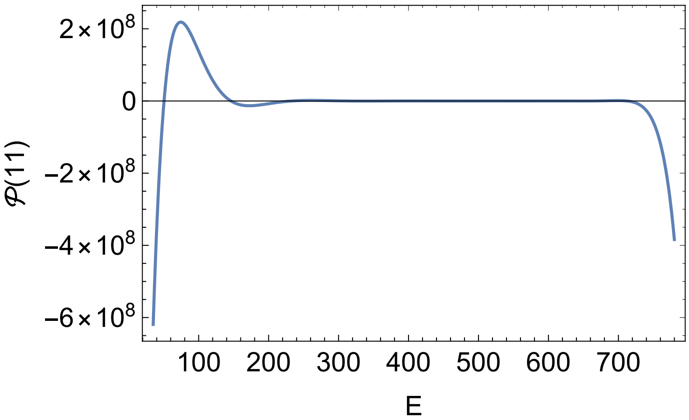
```
Solve[P1 == 0, Er] // FullSimplify
```

$\{\{Er \to 11 + (-2+\xi)\,\xi - 2\,l\,(2+\xi) - 2\sqrt{4 + l^2 + 4\,(-1+\xi)\,\xi - 4\,l\,(1+\xi)}\},$
$\{Er \to 11 + (-2+\xi)\,\xi - 2\,l\,(2+\xi) + 2\sqrt{4 + l^2 + 4\,(-1+\xi)\,\xi - 4\,l\,(1+\xi)}\}\}$

```
In[19]:= P11 = Pp[11] /. En[[1]] /. n → 11;
```

In[21]:=
```
Fig7 = Plot[P11 /. {α → 2, β → 1, ξ → 2}, {Er, 35, 780}, Frame → True,
  FrameLabel → {"E", "𝒫(11)"}, FrameStyle → Directive[14], PlotRange → All]
```

Out[21]=



In[22]:=
```
sol = Solve[{P11 /. {α → 2, β → 1, ξ → 2}} == 0, Er, WorkingPrecision → 100] // N
```

Out[22]=
```
{{Er → 50.5262}, {Er → 146.083}, {Er → 233.507}, {Er → 312.742},
 {Er → 383.69}, {Er → 446.18}, {Er → 499.874}, {Er → 544.126},
 {Er → 580.222}, {Er → 617.352}, {Er → 661.546}, {Er → 712.152}}
```

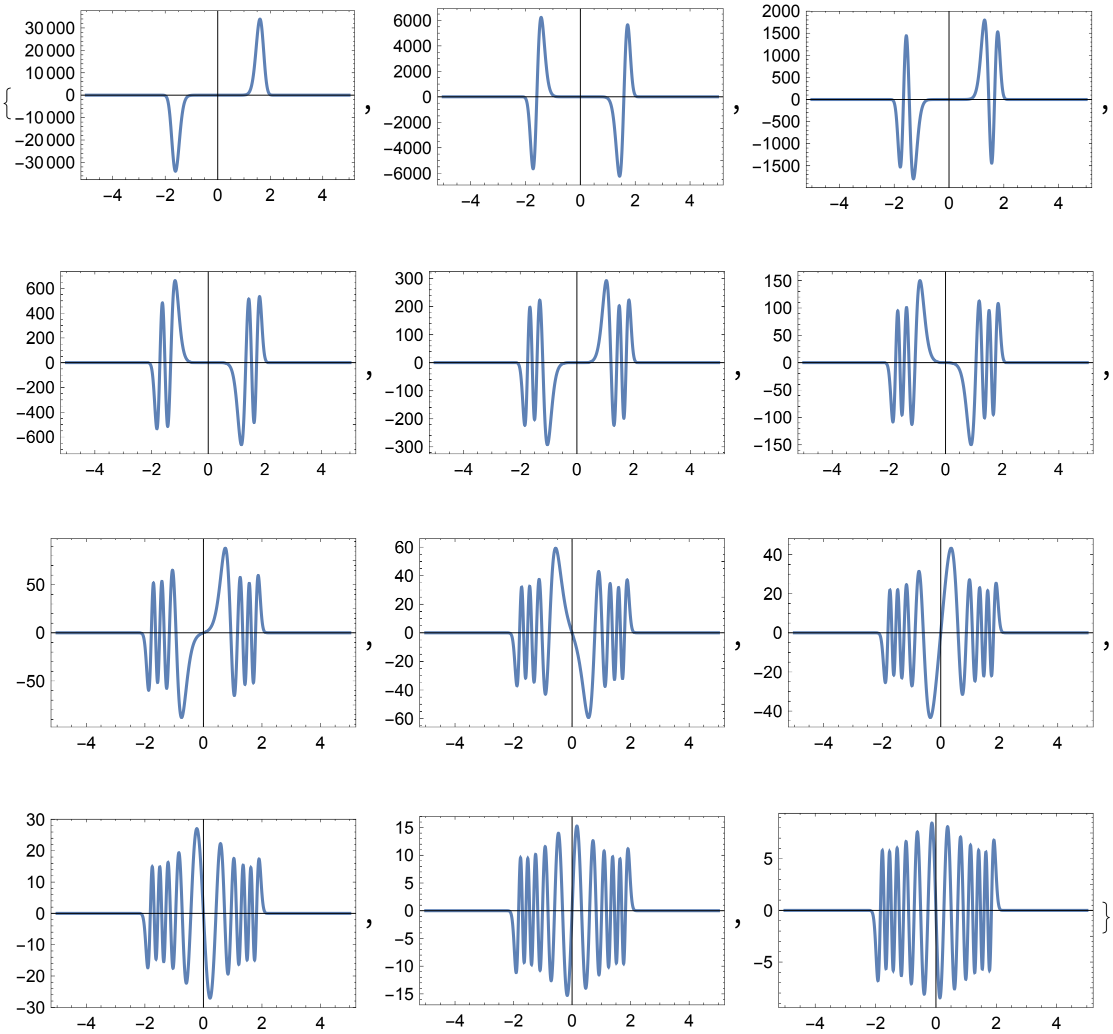
In[23]:=
```
mydata = Flatten[Cases[Fig7, Line[x__] → x, Infinity], 1];
Export["/Users/andreym/Desktop/Hatami_Fig7.dat",
  Join[{"Xx Yy"}, mydata], "Table"];
```

Wave function

In[25]:=
```
ψ[x_] = Exp[-ξ/2 Cosh[2 x]] Cosh[x]^α Sinh[x]^β S[11, z] /. En[[1]] /. z → Cosh[x]^2 /.
    {α → 2, β → 1, ξ → 2} /. n → 11;
```

In[26]:=
```
Table[Plot[ψ[x] /. sol[[i]], {x, -5, 5}, PlotRange → All, Frame → True], {i, 1, 12}]
```

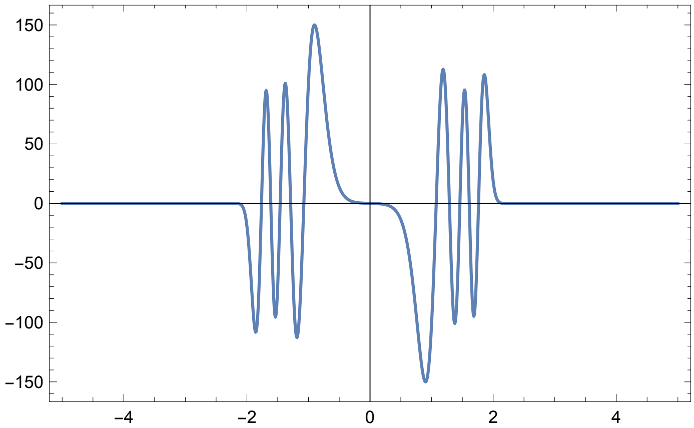
Out[26]=



In[27]:=
```
Wave[i_] := Plot[ψ[x] /. sol[[i]], {x, -5, 5}, PlotRange → All, Frame → True]
```

In[28]:=
```
Wave[6]
```

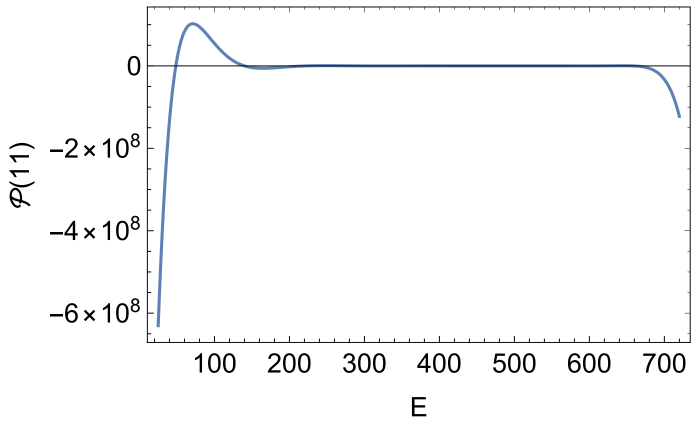
Out[28]=



In[29]:=
```
For[i = 1, i ≤ Length[sol], i = i + 1,
 wave = Flatten[Cases[Wave[i], Line[x__] → x, Infinity], 1];
 xs = StringJoin["x", ToString[i], " w", ToString[i]];
Export[StringJoin["/Users/andreym/Desktop/Hatami_Wave_DSGS_odd",
   ToString[i], ".dat"], Join[{xs}, wave], "Table"];
]
```

In[30]:=
```
Fig7s = Plot[P11 /. {α → 2, β → 0, ξ → 2}, {Er, 25, 720}, Frame → True,
  FrameLabel → {"E", "𝒫(11)"}, FrameStyle → Directive[14], PlotRange → All]
```

Out[30]=



In[31]:=
```
sol = Solve[{P11 /. {α → 2, β → 0, ξ → 2}} == 0, Er, WorkingPrecision → 100] // N
```

Out[31]=
```
{{Er → 48.5067}, {Er → 140.039}, {Er → 223.425}, {Er → 298.596},
 {Er → 365.435}, {Er → 423.725}, {Er → 472.987}, {Er → 511.035},
 {Er → 534.418}, {Er → 566.233}, {Er → 609.075}, {Er → 658.526}}
```

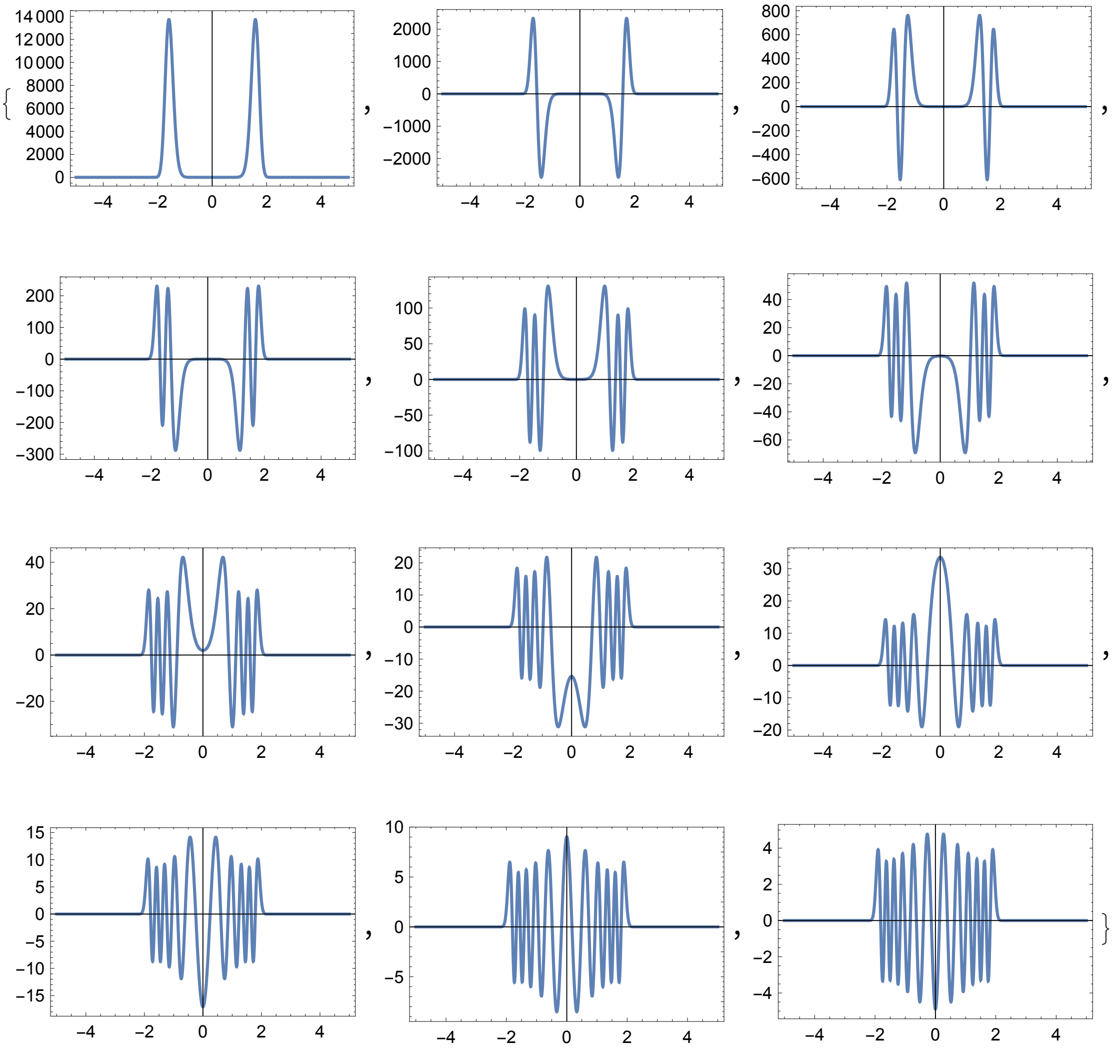
In[32]:=
```
mydata = Flatten[Cases[Fig7s, Line[x__] → x, Infinity], 1];
Export["/Users/andreym/Desktop/Hatami_Fig7s.dat",
  Join[{"Xx Yy"}, mydata], "Table"];
```

Wave function

In[34]:=
```
ψ[x_] = Exp[-ξ/2 Cosh[2 x]] Cosh[x]^α Sinh[x]^β S[11, z] /. z → Cosh[x]^2 /. En[[1]] /.
    {α → 2, β → 0, ξ → 2} /. n → 11;
```

In[35]:=
```
Table[Plot[ψ[x] /. sol[[i]], {x, -5, 5}, PlotRange → All, Frame → True], {i, 1, 12}]
```

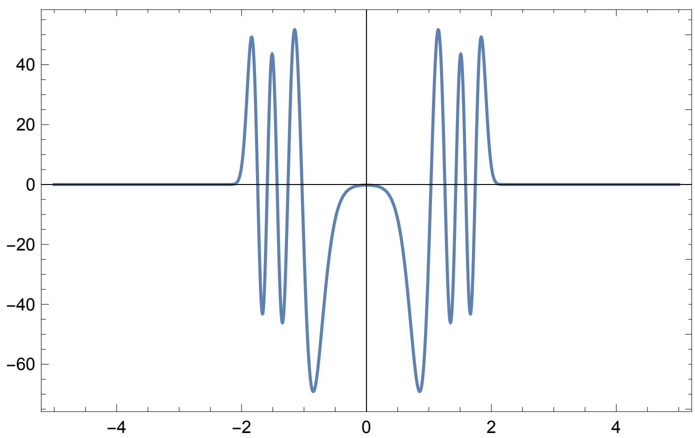
Out[35]=



In[36]:=
```
Wave[i_] := Plot[ψ[x] /. sol[[i]], {x, -5, 5}, PlotRange → All, Frame → True]
```

In[37]:=
```
Wave[6]
```

Out[37]=



In[38]:=
```
For[i = 1, i ≤ Length[sol], i = i + 1,
 wave = Flatten[Cases[Wave[i], Line[x__] → x, Infinity], 1];
 xs = StringJoin["x", ToString[i], " w", ToString[i]];
Export[StringJoin["/Users/andreym/Desktop/Hatami_Wave_DSGS_even",
   ToString[i], ".dat"], Join[{xs}, wave], "Table"];
]
```

Even vs Odd solutions (n=5, M=12)

In[39]:=
```
P5even = Pp[5] /. En[[1]] /. n → 5;
```

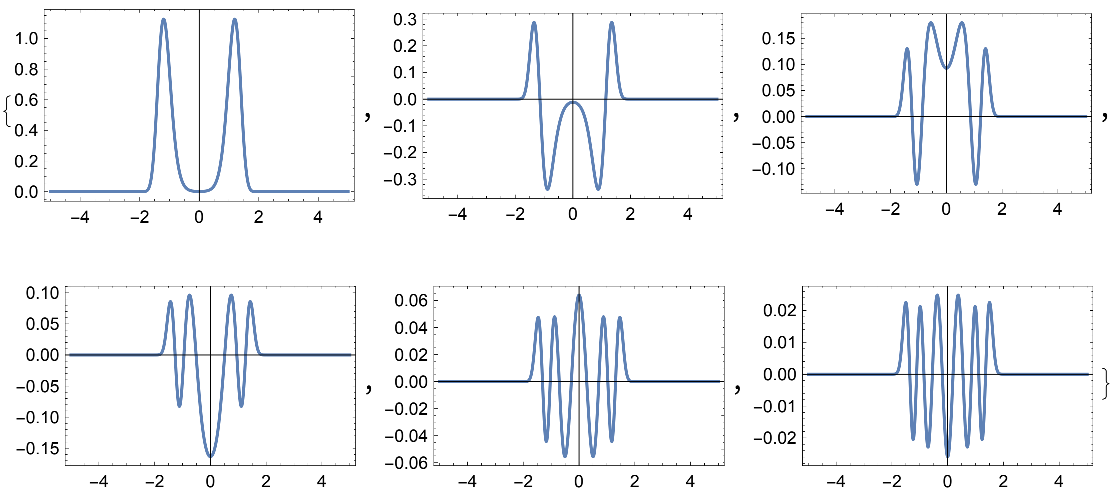
In[40]:=
```
sol5even =
 Solve[{P5even /. {α → 1, β → 0, ξ → 2}} == 0, Er, WorkingPrecision → 30] // Sort
```

Out[40]=
```
{{Er → 22.5949469112885432379698491502}, {Er → 61.3442522667606247136091581138},
 {Er → 89.8744853668145846480906179840}, {Er → 106.478216232079120401808861569},
 {Er → 131.616572062738164537324569480}, {Er → 166.091527160318962461196943703}}
```

In[41]:=
```
ψ[x_] = Exp[-ξ/2 Cosh[2 x]] Cosh[x]^α Sinh[x]^β S[5, z] /. z → Cosh[x]^2 /. En[[1]] /.
   {α → 1, β → 0, ξ → 2} /. n → 5;
```

In[42]:=
```
Table[Plot[ψ[x] /. sol5even[[i]],
  {x, -5, 5}, PlotRange → All, Frame → True], {i, 1, 6}]
```

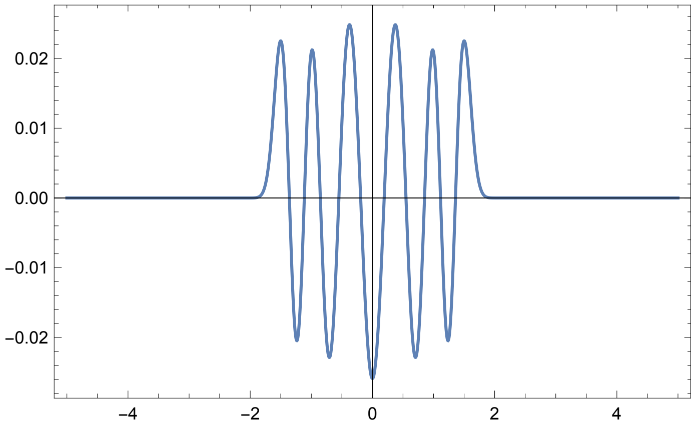
Out[42]=



In[43]:=
```
Wave[i_] := Plot[ψ[x] /. sol5even[[i]], {x, -5, 5}, PlotRange → All, Frame → True]
```

In[44]:=
```
Wave[6]
```

Out[44]=



In[45]:=
```
For[i = 1, i ≤ Length[sol5even], i = i + 1,
 wave = Flatten[Cases[Wave[i], Line[x__] → x, Infinity], 1];
 xs = StringJoin["x", ToString[i], " w", ToString[i]];
Export[StringJoin["/Users/andreym/Desktop/Hatami_WaveDSG_even",
   ToString[i], ".dat"], Join[{xs}, wave], "Table"];
]
```

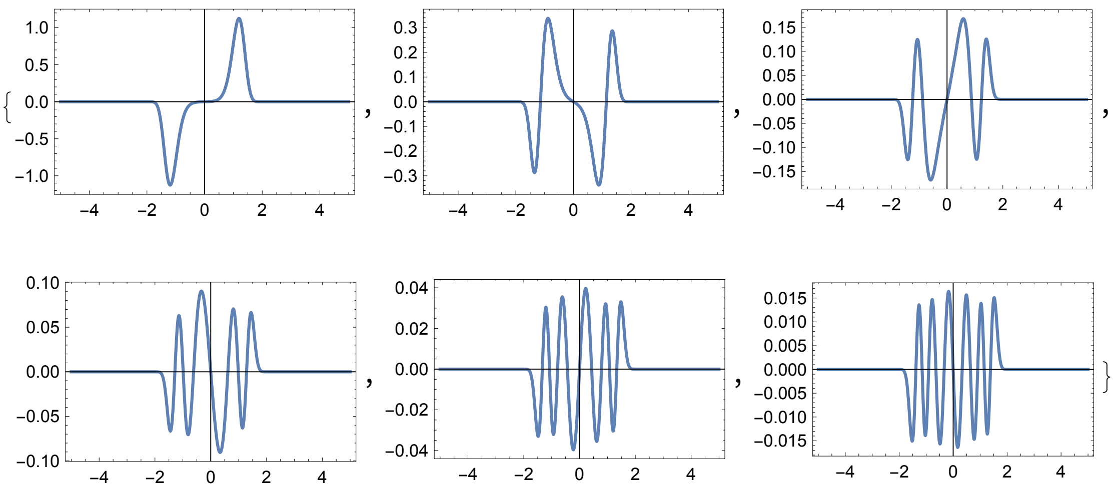
In[46]:=
```
sol5odd =
 Solve[{P5even /. {α → 0, β → 1, ξ → 2}} == 0, Er, WorkingPrecision → 30] // Sort
```

Out[46]=
```
{{Er → 22.5949681752431694437208058310}, {Er → 61.3580546887200459024743649307},
 {Er → 91.2808151738431821370117281477}, {Er → 117.007641477682941853484075105},
 {Er → 147.980766162041982471556899937}, {Er → 185.777754322468678191752126049}}
```

In[47]:=
```
ψ[x_] = Exp[-ξ/2 Cosh[2 x]] Cosh[x]^α Sinh[x]^β S[5, z] /. z → Cosh[x]^2 /. En[[1]] /.
   {α → 0, β → 1, ξ → 2} /. n → 5;
```

In[48]:=
```
Table[Plot[ψ[x] /. sol5odd[[i]],
  {x, -5, 5}, PlotRange → All, Frame → True], {i, 1, 6}]
```

Out[48]=



DSHG

```
a3 = 0;
a2 = -4;
a1 = 0;
b2 = 2 ξ;
b1 = 4 M - 8;
b0 = -2 ξ;
c1 = -2 ξ (M - 1);
c0 = -Er - 1 + 2 M + ξ^2;
```

```
F1[n]
```

$-2\,(-1+M)\,\xi + 2\,n\,\xi$

```
En = Solve[F1[n] == 0, M] // Simplify
```

$\{\{M \to 1+n\}\}$

```
F1[k] /. En[[1]] // FullSimplify
F0[k] - c0 /. En[[1]] // FullSimplify
Fm1[k] /. En[[1]] // FullSimplify
```

$2\,(k-n)\,\xi$

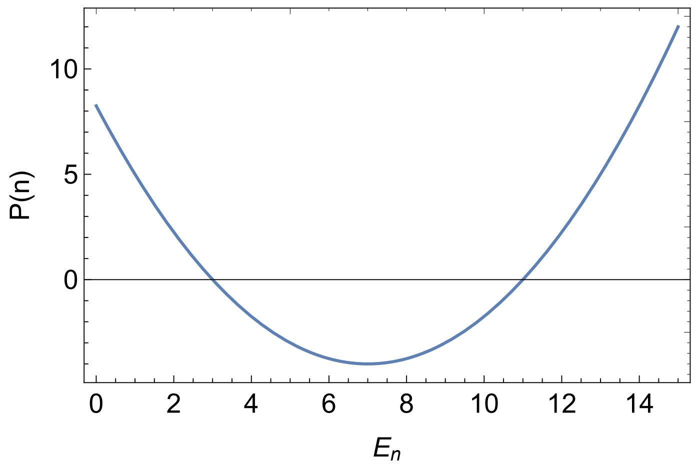
$4\,k\,(-k+n)$

$-2\,k\,\xi$

```
P1 = Pp[1] /. En[[1]] /. n → 1 /. {ξ → 2, M → 1} // FullSimplify
```

$\frac{1}{4}\,(-11+Er)\,(-3+Er)$

```
Plot[P1, {Er, 0, 15}, Frame → True,
 FrameLabel → {"En", "P(n)"}, FrameStyle → Directive[14]]
```

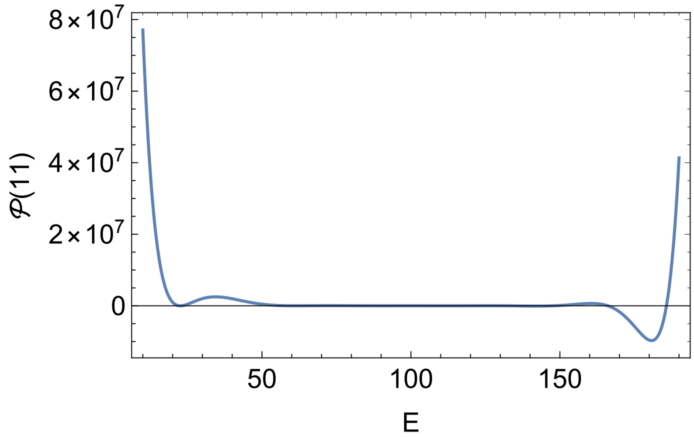



```
Solve[P1 == 0, Er] // N
```

$\{\{Er \to 3.\}, \{Er \to 11.\}\}$

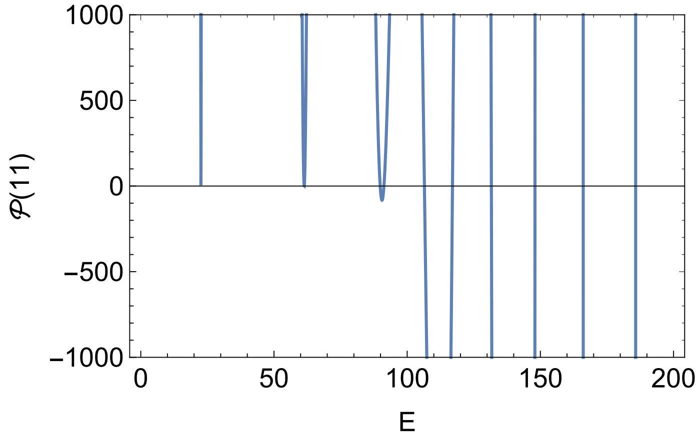
```
P11 = Pp[11] /. En[[1]] /. n → 11 /. {ξ → 2, M → 12};
```

```
Fig9 = Plot[P11, {Er, 10, 190}, Frame → True,
  FrameLabel → {"E", "𝒫(11)"}, FrameStyle → Directive[14], PlotRange → All]
```



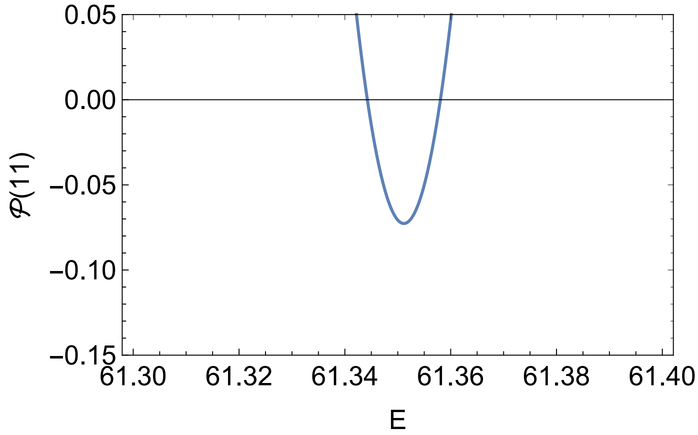

```
mydata = Flatten[Cases[Fig9, Line[x__] → x, Infinity], 1];
Export["/Users/andreym/Desktop/Hatami_Fig9.dat",
  Join[{"Xx Yy"}, mydata], "Table"];
```

```
Fig9b = Plot[P11, {Er, 10, 200}, Frame → True, FrameLabel → {"E", "𝒫(11)"},
  FrameStyle → Directive[14], PlotRange → {-1000, 1000}]
```



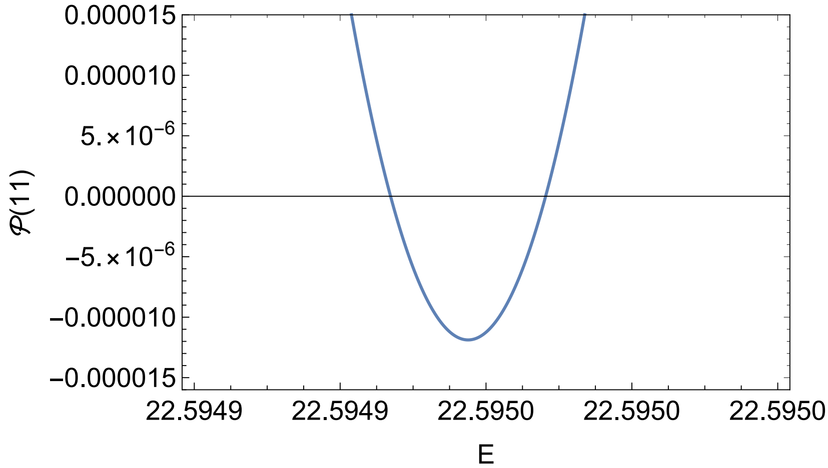

```
mydata = Flatten[Cases[Fig9b, Line[x__] → x, Infinity], 1];
Export["/Users/andreym/Desktop/Hatami_Fig9b.dat",
  Join[{"Xx Yy"}, mydata], "Table"];
```

```
Fig9c = Plot[P11, {Er, 61.3, 61.4}, Frame → True, FrameLabel → {"E", "𝒫(11)"},
  FrameStyle → Directive[14], PlotRange → {-0.15, 0.05}, PlotPoints → 10 000]
```




```
mydata = Flatten[Cases[Fig9c, Line[x__] → x, Infinity], 1];
Export["/Users/andreym/Desktop/Hatami_Fig9c.dat",
  Join[{"Xx Yy"}, mydata], "Table"];
```

```
Fig9d = Plot[P11, {Er, 22.59492, 22.595}, Frame → True,
  FrameLabel → {"E", "𝒫(11)"}, FrameStyle → Directive[14],
  PlotRange → {-0.000016, 0.000015}, PlotPoints → 10 000]
```



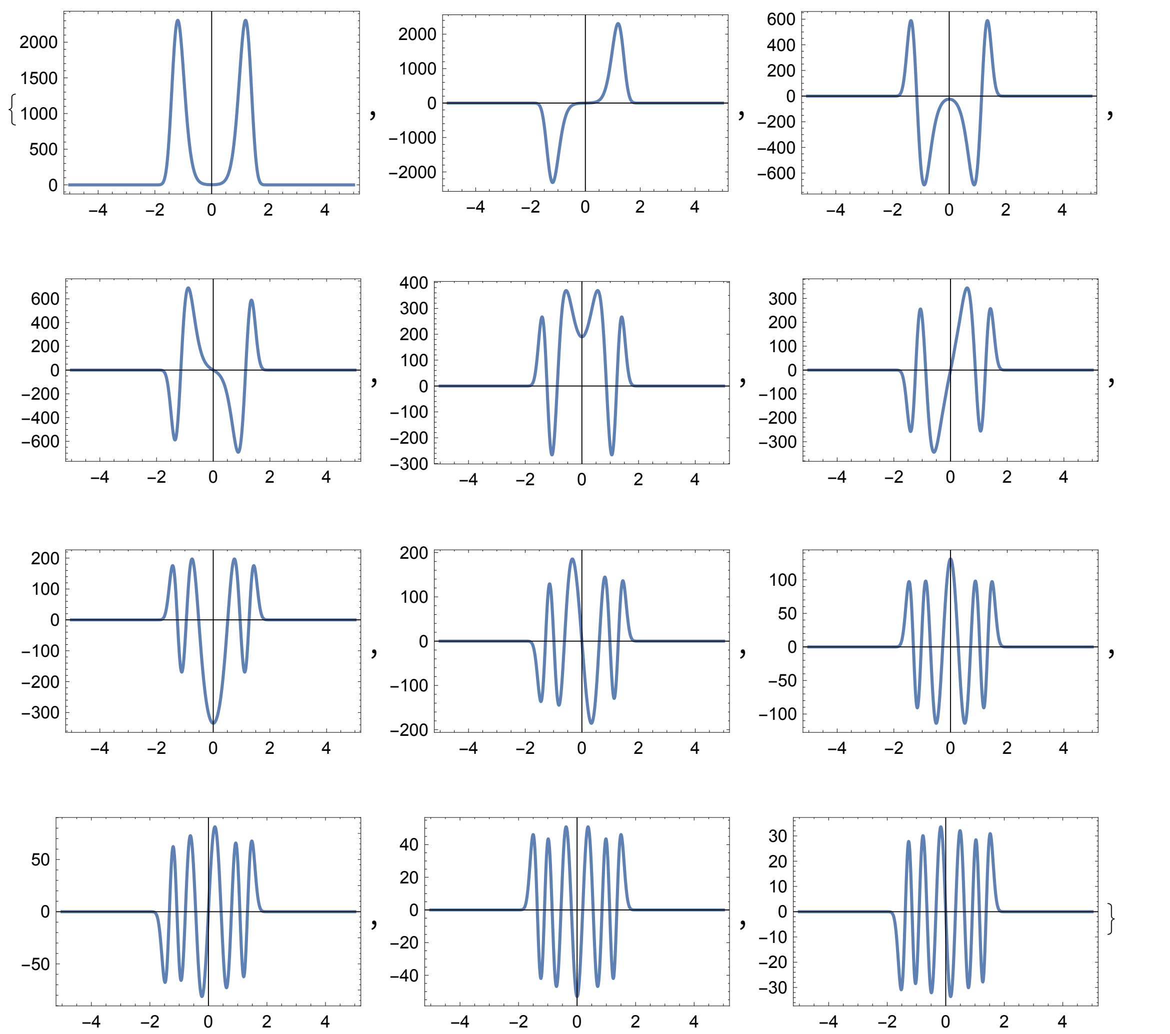

```
mydata = Flatten[Cases[Fig9d, Line[x__] → x, Infinity], 1];
Export["/Users/andreym/Desktop/Hatami_Fig9d.dat",
  Join[{"Xx Yy"}, mydata], "Table"];
```

```
sol = Solve[P11 == 0, Er, WorkingPrecision → 50] // Sort
```

```
{{Er → 22.594946911288543237969849150194478877162428648935},
 {Er → 22.594968175243169443720805831002703368142530721543},
 {Er → 61.344252266760624713609158113805179637839853870782},
 {Er → 61.358054688720045902474364930674597041600517819963},
 {Er → 89.874485366814584648090617984048083900706620446105},
 {Er → 91.280815173843182137011728147669597947275412206919},
 {Er → 106.47821623207912040180886156882739775904116437044},
 {Er → 117.00764147768294185348407510486201669866644077000},
 {Er → 131.61657206273816453732456948000202085759229386352},
 {Er → 147.98076616204198247155689993679781759268396718185},
 {Er → 166.09152716031896246119694370312283896765763880022},
 {Er → 185.77775432246867819175212604899326735163113129972}}
```

$$\psi[x\_] = z^{\frac{1-M}{2}}\,\mathrm{Exp}\left[-\frac{\xi}{4}\left(z+\frac{1}{z}\right)\right] S[11, z] \;/.\; z \to \mathrm{Exp}[2\,x] \;/.\; En[[1]] \;/.\; \{\xi \to 2\} \;/.\; n \to 11;$$

```
Table[Plot[ψ[x] /. sol[[i]], {x, -5, 5}, PlotRange → All, Frame → True], {i, 1, 12}]
```



Chen et al modified Manning potential with three parameters

$$a3 = 1;$$

$$a2 = -2 - \frac{1}{g};$$

$$a1 = 1 + \frac{1}{g};$$

$$b2 = 2\,\lambda1 + 2\,\lambda2 + 1;$$

$$b1 = -\left(2\,\lambda1 + 2\,\lambda2 + \frac{3}{2} + \frac{2\,\lambda1 + 1}{g}\right);$$

$$b0 = \frac{1+g}{2\,g};$$

$$c1 = (\lambda1 + \lambda2)^2 + \frac{Er}{4};$$

$$c0 = -\frac{1+g}{4\,g}\left(2\,\lambda1 + \frac{2\,\lambda2\,g - V2}{1+g} - V1 - \frac{V3}{(1+g)^2} + Er\right);$$

$$\lambda = \left\{\lambda1 \to \frac{1}{4}\left(1 + \sqrt{1 - 4\,V1}\right),\ \lambda2 \to \frac{1}{2}\left(1 - \sqrt{1 + \frac{V3}{1+g}}\right)\right\};$$

```
b1 // FullSimplify
```

$$-\frac{2 + 4\,\lambda1 + g\,(3 + 4\,\lambda1 + 4\,\lambda2)}{2\,g}$$

```
F1[n]
```

$$\frac{Er}{4} + (-1+n)\,n + (\lambda1 + \lambda2)^2 + n\,(1 + 2\,\lambda1 + 2\,\lambda2)$$

```
En = Solve[F1[n] == 0, Er] // Simplify
```

$$\{\{Er \to -4\,(n + \lambda1 + \lambda2)^2\}\}$$

```
F1[k] /. En[[1]] // FullSimplify
F0[k] - c0 /. En[[1]] // FullSimplify
Fm1[k] /. En[[1]] // FullSimplify
```

$$(k - n)\,(k + n + 2\,(\lambda1 + \lambda2))$$

$$-\frac{k\,(2\,(k + 2\,\lambda1) + g\,(-1 + 4\,k + 4\,\lambda1 + 4\,\lambda2))}{2\,g}$$

$$\frac{(1+g)\,k\,(-1 + 2\,k)}{2\,g}$$

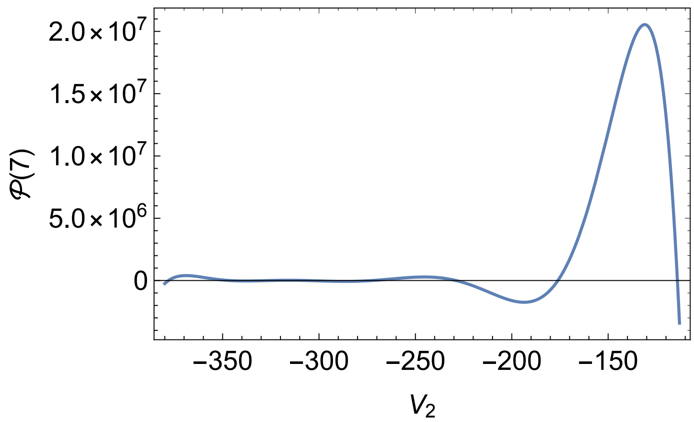
```
c0 /. En[[1]] // FullSimplify
```

$$-\frac{(1+g)\left(-V1 - \frac{V3}{(1+g)^2} + 2\,\lambda1 - (n + \lambda1 + \lambda2)^2 - \frac{V2 - 2\,g\,\lambda2}{1+g}\right)}{4\,g}$$

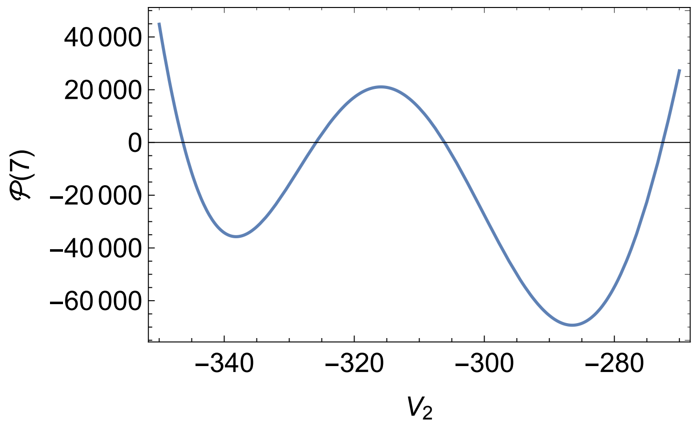
```
λ1 + λ2 + n /. λ /. {V1 → 0.09, g → 0.25} /. n → 7
```

$$7.45 + \frac{1}{2}\left(1 - \sqrt{1 + 0.8\,V3}\right)$$

```
Solve[(λ1 + λ2 + n /. λ /. {V1 → 0.09, g → 0.25} /. n → 7) == 0, V3]
{{V3 → 314.763}}

P7 = Pp[7] /. En[[1]] /. λ /. n → 7 /. {V1 → 0.09, V3 → 400, g → 0.25};

sol = NSolve[P7 == 0, V2]
{{V2 → -378.075}, {V2 → -346.334}, {V2 → -325.892}, {V2 → -306.113},
 {V2 → -272.536}, {V2 → -228.953}, {V2 → -176.075}, {V2 → -114.078}}

Fig10 = Plot[Re[P7], {V2, -380, -113}, Frame → True,
  FrameLabel → {"V2", "𝒫(7)"}, FrameStyle → Directive[14], PlotRange → All]
```




```
Plot[Re[P7], {V2, -350, -270}, Frame → True,
 FrameLabel → {"V2", "𝒫(7)"}, FrameStyle → Directive[14]]
```




```
mydata = Flatten[Cases[Fig10, Line[x__] → x, Infinity], 1];
Export["/Users/andreym/Desktop/Hatami_Fig10.dat",
  Join[{"Xx Yy"}, mydata], "Table"];
```

Wave function

```
λ1 + λ2 + n /. λ /. En[[1]] /. {V1 → 0.09, V3 → 400, g → 1/4} /. n → 7
```

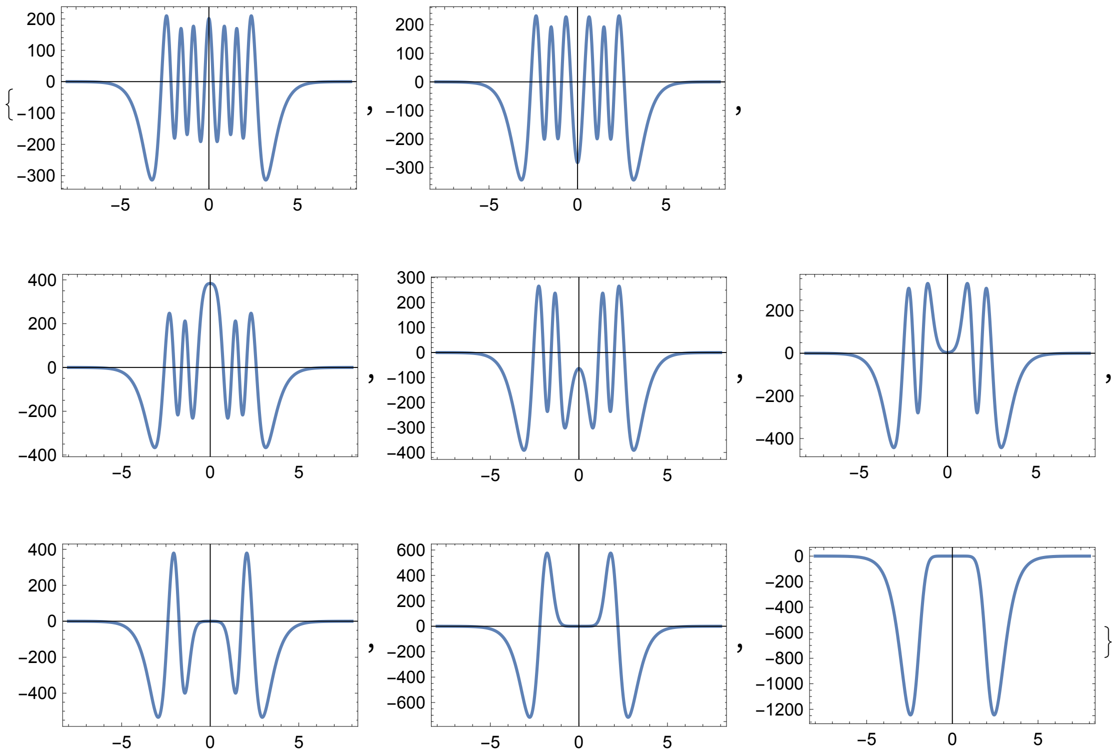
```
-1.00824
```

```
Solve[(λ1 + λ2 + n /. λ /. En[[1]] /. {V1 → 0.09, g → 1/4} /. n → 7) == 0, V3]
```

```
{{V3 → 314.763}}
```

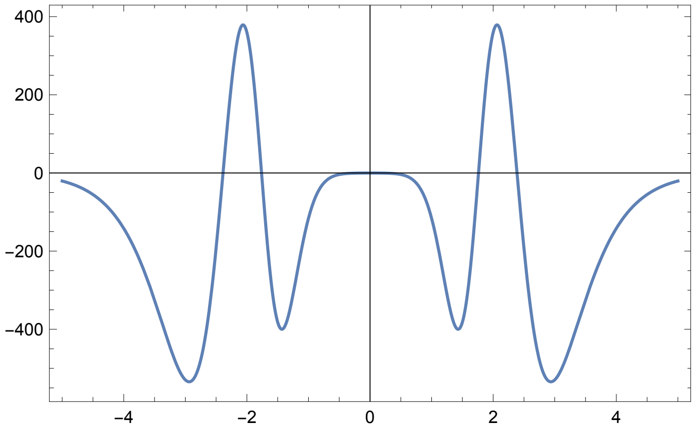
```
ψ[x_] = Cosh[x]^(2 λ1) (1 + g Cosh[x]^2)^λ2 S[7, z] /. z → -Sinh[x]^2 /. En[[1]] /. λ /.
    {V1 → 0.09, V3 → 400, g → 1/4} /. n → 7;
```

```
Table[Plot[ψ[x] /. λ /. {V1 → 0.09, V3 → 400, g → 1/4} /. n → 7 /. sol[[i]],
  {x, -8, 8}, PlotRange → All, Frame → True], {i, 1, 8}]
```




```
Wave[i_] := Plot[ψ[x] /. sol[[i]], {x, -5, 5}, PlotRange → All, Frame → True]
```

```
Wave[6]
```




```
For[i = 1, i ≤ Length[sol], i = i + 1,
 wave = Flatten[Cases[Wave[i], Line[x__] → x, Infinity], 1];
 xs = StringJoin["x", ToString[i], " w", ToString[i]];
Export[StringJoin["/Users/andreym/Desktop/Hatami_Wave6_s",
   ToString[i], ".dat"], Join[{xs}, wave], "Table"];
]
```

Odd parity solutions

$$a3 = 1;$$
$$a2 = -2 - \frac{1}{g};$$
$$a1 = 1 + \frac{1}{g};$$
$$b2 = 2\,(\lambda1 + \lambda2 + 1);$$
$$b1 = -\left(2\,\lambda1 + 2\,\lambda2 + \frac{7}{2} + \frac{2\,(\lambda1 + 1)}{g}\right);$$
$$b0 = \frac{3\,(1 + g)}{2\,g};$$
$$c1 = (\lambda1 + \lambda2)\,(\lambda1 + \lambda2 + 1) + \frac{Er + 1}{4};$$
$$c0 = -\frac{1 + g}{4\,g}\left(6\,\lambda1 + 4\,\lambda2 + 1 + \frac{2\,\lambda2\,g - V2}{1 + g} - V1 - \frac{V3}{(1 + g)^2} + Er\right) + \frac{\lambda2}{g};$$
$$\lambda = \left\{\lambda1 \to \frac{1}{4}\left(1 + \sqrt{1 - 4\,V1}\right),\ \lambda2 \to \frac{1}{2}\left(1 - \sqrt{1 + \frac{V3}{1 + g}}\right)\right\};$$

```
F1[n]
```

$$\frac{1 + Er}{4} + (-1 + n)\,n + 2\,n\,(1 + \lambda1 + \lambda2) + (\lambda1 + \lambda2)\,(1 + \lambda1 + \lambda2)$$

```
En = Solve[F1[n] == 0, Er] // Simplify
```

$$\left\{\left\{Er \to -(1 + 2\,n + 2\,\lambda1 + 2\,\lambda2)^2\right\}\right\}$$

```
F1[k] /. En[[1]] // FullSimplify
F0[k] - c0 /. En[[1]] // FullSimplify
Fm1[k] /. En[[1]] // FullSimplify
```

$(k - n)\ (1 + k + n + 2\,\lambda 1 + 2\,\lambda 2)$

$-\frac{k\,(2\,(1 + k + 2\,\lambda 1) + g\,(3 + 4\,k + 4\,\lambda 1 + 4\,\lambda 2))}{2\,g}$

$\frac{(1 + g)\ k\ (1 + 2\,k)}{2\,g}$

```
c0 /. En[[1]] // FullSimplify
```

$\frac{1}{4\,g}\left(4\,\lambda 2 - (1 + g)\ \left(1 - V1 - \frac{V3}{(1 + g)^2} + 6\,\lambda 1 + 4\,\lambda 2 - (1 + 2\,n + 2\,\lambda 1 + 2\,\lambda 2)^2 - \frac{V2 - 2\,g\,\lambda 2}{1 + g}\right)\right)$

$\lambda 1 + \lambda 2 + n + \frac{1}{2}$ /. λ /. {V1 → 0.09, V2 → 10, g → 0.25}

$0.95 + n + \frac{1}{2}\left(1 - \sqrt{1 + 0.8\,V3}\right)$

Solve[$\left(\lambda 1 + \lambda 2 + n + \frac{1}{2}\right.$ /. λ /. {V1 → 0.09, V2 → 10, g → 0.25, n → 7}) == 0, V3]

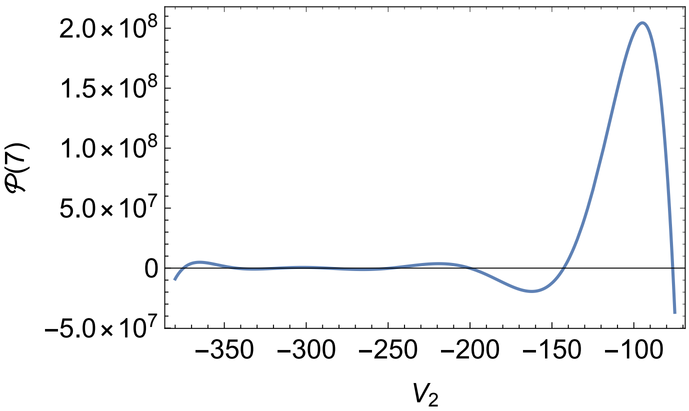
{{V3 → 355.763}}

```
P7 = Pp[7] /. En[[1]] /. λ /. n → 7 /. {V1 → 0.09, V3 → 400, g → 0.25};
```

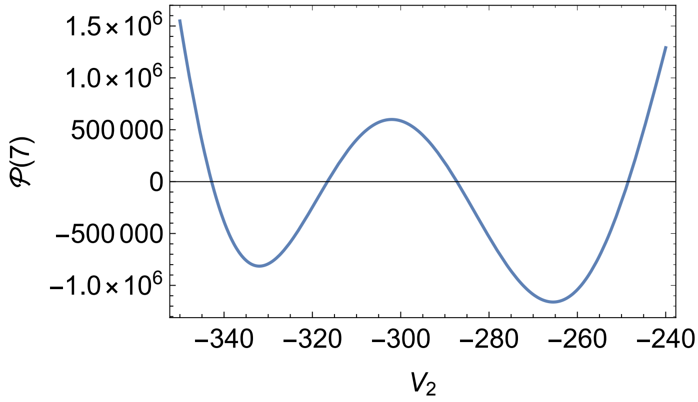
```
sol = Solve[P7 == 0, V2]
```

{{V2 → -374.929}, {V2 → -342.812}, {V2 → -316.597}, {V2 → -287.269},
{V2 → -248.489}, {V2 → -200.236}, {V2 → -142.792}, {V2 → -76.2691}}

```
Fig11 = Plot[Re[P7], {V2, -380, -75}, Frame → True,
  FrameLabel → {"V2", "𝒫(7)"}, FrameStyle → Directive[14], PlotRange → All]
```



```
Plot[Re[P7], {V2, -350, -240}, Frame → True,
 FrameLabel → {"V₂", "𝒫(7)"}, FrameStyle → Directive[14]]
```




```
mydata = Flatten[Cases[Fig11, Line[x__] → x, Infinity], 1];
Export["/Users/andreym/Desktop/Hatami_Fig11.dat",
  Join[{"Xx Yy"}, mydata], "Table"];
```

Wave function

$$\lambda 1 + \lambda 2 + n + \frac{1}{2} \;/.\; \lambda \;/.\; \text{En}[[1]] \;/.\; \left\{V1 \to 0.09,\ V3 \to 400,\ g \to \frac{1}{4}\right\} \;/.\; n \to 7$$

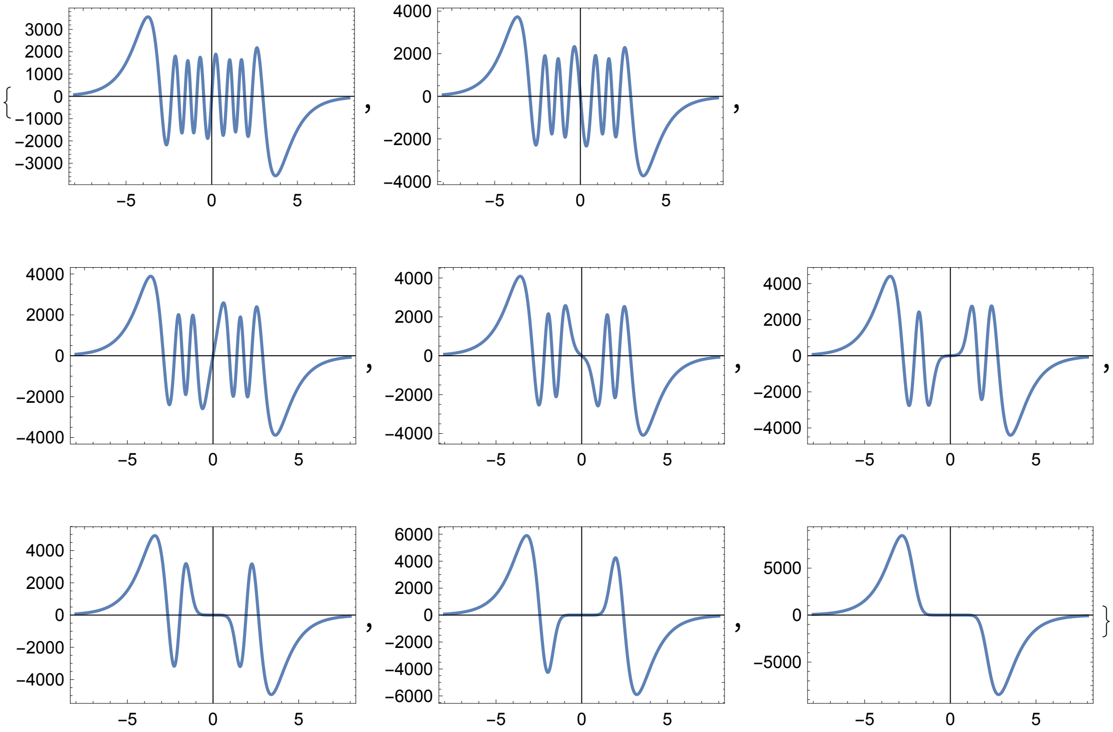
-0.508236

$$\text{Solve}\left[\left(\lambda 1 + \lambda 2 + n + \frac{1}{2} \;/.\; \lambda \;/.\; \text{En}[[1]] \;/.\; \left\{V1 \to 0.09,\ g \to \frac{1}{4}\right\} \;/.\; n \to 7\right) == 0,\ V3\right]$$

{{V3 → 355.763}}

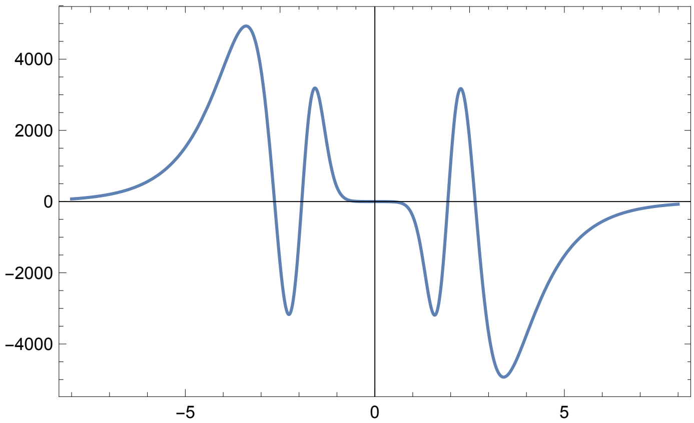
$$\psi[x\_] = \text{Cosh}[x]^{2\lambda 1} \left(1 + g\,\text{Cosh}[x]^2\right)^{\lambda 2} \text{Sinh}[x]\ S[7, z] \;/.\; z \to -\text{Sinh}[x]^2 \;/.\; \text{En}[[1]] \;/.\; \lambda \;/.\; \left\{V1 \to 0.09,\ V3 \to 400,\ g \to \frac{1}{4}\right\} \;/.\; n \to 7;$$

```
Table[Plot[ψ[x] /. λ /. {V1 → 0.09, V3 → 400, g → 1/4} /. n → 7 /. sol[[i]],
  {x, -8, 8}, PlotRange → All, Frame → True], {i, 1, 8}]
```



```
Wave[i_] := Plot[ψ[x] /. sol[[i]], {x, -8, 8}, PlotRange → All, Frame → True]
```

```
Wave[6]
```



```
For[i = 1, i ≤ Length[sol], i = i + 1,
 wave = Flatten[Cases[Wave[i], Line[x__] → x, Infinity], 1];
 xs = StringJoin["x", ToString[i], " w", ToString[i]];
Export[StringJoin["/Users/andreym/Desktop/Hatami_Wave7_s",
   ToString[i], ".dat"], Join[{xs}, wave], "Table"];
]
```